

MIDTERM 2, Math 8601, Fall 2006

Solve problem 1, and ONE of the problems 2 and 3. Clearly indicate your choice. Good luck!

Problem 1. (25 points: 5 points each, 1 point for correct answer, 4 points for correct explanation) Let E be a Banach space. True or False?

- (i) Every linear function $T : E \rightarrow E$ is Gateaux differentiable at some $x_0 \in E$.

False: T has all directional derivatives: $(\nabla_v T)(x_0) = T(v)$ and so it is Gateaux differentiable (at any point x_0) iff it is continuous. We know that if E is infinitely dimensional, there exists discontinuous linear functions.

- (ii) Any continuous function $f : B \rightarrow \bar{B}$ (where B is the open unit ball in E and \bar{B} is the closed unit ball in E) can be extended to a continuous function $\tilde{f} : E \rightarrow \bar{B}$.

False: Let $E = \mathbf{R}$, and define $f(x) = \sin(1/(x+1))$. Then f cannot be extended to a continuous function on \mathbf{R} because it does not have a limit as $x \rightarrow -1$.

- (iii) Any continuous function $f : \bar{B} \rightarrow B$ (where B is the open unit ball in E and \bar{B} is the closed unit ball in E) can be extended to a continuous function $\tilde{f} : E \rightarrow B$.

True: By Dugundji's theorem B is an extensor as a convex subset of a normed space E .

- (iv) Every functional $T \in c^*$ is of the form $T(\{x_i\}_{i=1}^\infty) = \sum_{i=1}^\infty \alpha_i x_i$, for some sequence $\alpha = \{\alpha_i\}_{i=1}^\infty$ belonging to l_1 .

False: $T(\{x\}) = \lim x_i$ cannot be represented this way. Because otherwise we would have $1 = T(1, 1, 1, \dots) = \sum_{i=1}^\infty \alpha_i$, and $1 = T(0, 1, 1, \dots) = \sum_{i=2}^\infty \alpha_i$, and $1 = T(0, 0, 1, \dots) = \sum_{i=3}^\infty \alpha_i, \dots$ etc. So there would be $\alpha_i = 0$ for every i which contradicts $\sum \alpha_i = 1$.

- (v) There is no injective function $f \in \mathcal{C}^1(\mathbf{R}^2, \mathbf{R})$.

True: Take two continuous curves $\gamma_1, \gamma_2 : [0, 1] \rightarrow \mathbf{R}^2$ joining some points $a \neq b$ in \mathbf{R}^2 : $\gamma_1(0) = \gamma_2(0) = a$ and \mathbf{R}^2 : $\gamma_1(1) = \gamma_2(1) = b$, and not intersection anywhere (but the endpoints). By mean value theorem (on the real line!), the functions $f \circ \gamma_1$ and $f \circ \gamma_2$ must attain all values inside the interval $[f(a), f(b)]$. This contradicts the injectivity of f .

[We see that we used only the continuity of f]

Problem 2. (25 points) Let Λ be a metric space and E a Banach space. Let $\Phi : \Lambda \times E \longrightarrow E$ be a map such that:

$$(1) \quad \exists \kappa < 1 \quad \forall x, y \in E \quad \forall \lambda \in \Lambda \quad \|\Phi(\lambda, x) - \Phi(\lambda, y)\| \leq \kappa \|x - y\|,$$

$$(2) \quad \forall \lambda_0 \in \Lambda \quad \forall y \in E \quad \lim_{\lambda \rightarrow \lambda_0} \Phi(\lambda, y) = \Phi(\lambda_0, y).$$

Prove that:

- (i) for every $\lambda \in \Lambda$ there exists a unique $x(\lambda) \in E$ such that $x(\lambda) = \Phi(\lambda, x(\lambda))$,
- (ii) the function $\lambda \mapsto x(\lambda)$ is continuous.

Since each function $\Phi_\lambda = \Phi(\lambda, \cdot) : E \longrightarrow E$ is Lipschitz continuous with Lipschitz constant $\kappa < 1$, (i) is clear by the Banach contraction principle. Moreover:

$$(3) \quad \forall \lambda \in \Lambda \quad \|y - x(\lambda)\| \leq \frac{1}{1 - \kappa} \|y - \Phi(\lambda, y)\|.$$

Indeed, we have (as in the proof of the Banach theorem):

$$\|\Phi_\lambda^{(n)}(y) - y\| \leq \sum_{j=0}^{n-1} \|\Phi_\lambda^{(j+1)}(y) - \Phi_\lambda^{(j)}(y)\| \leq \sum_{j=0}^{n-1} \kappa^j \|\Phi_\lambda(y) - y\| \leq \frac{1}{1 - \kappa} \|\Phi_\lambda(y) - y\|.$$

Since $\Phi_\lambda^{(n)}(y) \rightarrow x(\lambda)$ as $n \rightarrow \infty$, we conclude (3). Use now (3) with $y = x(\lambda_0)$:

$$\limsup_{\lambda \rightarrow \lambda_0} \|x(\lambda_0) - x(\lambda)\| \leq \frac{1}{1 - \kappa} \limsup_{\lambda \rightarrow \lambda_0} \|\Phi(\lambda_0, x(\lambda_0)) - \Phi(\lambda, x(\lambda_0))\| = 0.$$

A shorter proof of (ii) goes as follows. By (1) we have:

$$\begin{aligned} \|x(\lambda) - x(\lambda_0)\| &\leq \|\Phi(\lambda, x(\lambda)) - \Phi(\lambda, x(\lambda_0))\| + \|\Phi(\lambda, x(\lambda_0)) - \Phi(\lambda_0, x(\lambda_0))\| \\ &\leq \kappa \|x(\lambda) - x(\lambda_0)\| + \|\Phi(\lambda, x(\lambda_0)) - \Phi(\lambda_0, x(\lambda_0))\|. \end{aligned}$$

Therefore, using the above and (2):

$$\|x(\lambda) - x(\lambda_0)\| \leq \frac{1}{1 - \kappa} \|\Phi(\lambda, x(\lambda_0)) - \Phi(\lambda_0, x(\lambda_0))\| \rightarrow 0 \quad \text{as } \lambda \rightarrow \lambda_0.$$

Problem 3. (25 points) Let $\{f_n\}_{n=1}^\infty$ be a sequence of \mathcal{C}^1 maps from an open subset U of a Banach space E into a Banach space F . Assume that $\{f_n\}$ converge pointwise to a map $f : U \longrightarrow F$ and that the sequence of derivatives $\{f'_n\}$ converges uniformly to a mapping $g : U \longrightarrow \mathcal{L}(E, F)$. Prove that f is \mathcal{C}^1 and that $f' = g$.

Fix $x_0 \in E$. Let $x \in B_\delta(x_0) \subset U$. We have:

$$\begin{aligned} \frac{\|f(x) - f(x_0) - g(x_0)(x - x_0)\|}{\|x - x_0\|} &\leq \frac{\|f_n(x) - f(x) - [f_n(x_0) - f(x_0)]\|}{\|x - x_0\|} \\ &+ \frac{\|f_n(x) - f_n(x_0) - f'_n(x_0)(x - x_0)\|}{\|x - x_0\|} + \frac{\|f'_n(x_0)(x - x_0) - g(x_0)(x - x_0)\|}{\|x - x_0\|} \\ &\longrightarrow 0, \quad \text{as } \|x - x_0\| \rightarrow 0. \end{aligned}$$

The convergence of the second term is clear. The third term is bounded by $\|f'_n(x_0) - g(x_0)\|$ and so it converges to 0 by the assumption. To deal with the first term, apply the mean value theorem to the function $f_n - f_m$:

$$\|f_n(x) - f_m(x) - [f_n(x_0) - f_m(x_0)]\| \leq \|x - x_0\| \cdot \sup_{y \in B_\delta(x_0)} \|f'_n(y) - f'_m(y)\|.$$

By the uniform convergence of derivatives, the last "sup" converges to 0 as $\delta \rightarrow 0$. Now pass $m \rightarrow \infty$, to conclude the convergence of the first term. Notice that g is continuous as a uniform limit of continuous functions, so f is \mathcal{C}^1 .