MIDTERM 3, Math 8601, Fall 2006

Solve problem 1, and ONE of the problems 2 and 3. Clearly indicate your choice. Good luck!

Problem 1. (25 points: 5 points each, 1 point for correct answer, 4 points for correct explanation) All statements refer to the Lebesgue measure in $\mathbf{R}^n$. True or False?

(i) A set $A \subset \mathbf{R}^n$ is measurable if and only if the set $A \times \mathbf{R} \subset \mathbf{R}^{n+1}$ is measurable.

True: If $A \in \mathcal{L}_n$ *then* $A = F \cup N$, *and* F *is a* F_σ *set,* $\mu(N) = 0$. *Then* $A \times \mathbf{R} = (F \times \mathbf{R}) \cup (N \times \mathbf{R})$, *again an* F_σ *set and a set of measure* 0. *Conversly, if* $A \times \mathbf{R} \in \mathcal{L}_{n+1}$ *then its characteristic function is measurable, and the conclusion follows by Fubini's theorem.*

(ii) The boundary of a measurable set $A \subset \mathbf{R}^n$ must have measure 0.

False: The boundary of $A = Q \subset \mathbf{R}$ *is the whole* $\mathbf{R}$.

(iii) The Vitali set contains a measurable set of positive measure.

False: If $A \subset V$ *and* $\mu(A) > 0$ *then consider the sets* $\{A + q\}_{q \in Q}$, *as in the proof of nonmeasurability of* V. *We have:* $\mu(A + q) = \mu(A)$, *for all* $q \in Q$. *These sets are pairwise disjoint and contained in* $[0, 1]$, *so here comes the contradiction:*

$$1 = \mu([0, 1]) \geq \sum_{q \in Q} \mu(A + q) = \sum_{q \in Q} \mu(A) = \infty.$$

(iv) A function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is measurable if and only if the set $f^{-1}(\{\alpha\})$ is measurable, for every $\alpha \in \mathbf{R}$.

False: Take a nonmeasurable set $V \subset [0, 1]$ *and define* $f : \mathbf{R} \longrightarrow \mathbf{R}$*:*

$$f(x) = \begin{cases} x & \text{if } x \in V \\ -x & \text{if } x \notin V \end{cases}$$

Then $f^{-1}(\{\alpha\})$ *contains at most two points for each* α, *and so it is measurable, but* f *is clearly not measurable.*

(v) A function $f : \mathbf{R} \longrightarrow \mathbf{R}$ is measurable if and only if its graph has measure zero in $\mathbf{R}^2$.

False: The graph of the characteristic function of any nonmeasurable set (such function must be nonmeasurable) has measure 0, *as it is contained in two lines.*

Problem 2. (25 points)
Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n=1}^{\infty}, f : X \longrightarrow \mathbf{R}$ be a sequence of μ-measurable functions such that for every $\epsilon > 0$:

$$\lim_{n\to\infty} \mu(\{x \in X;\ |f_n(x) - f(x)| > \epsilon\}) = 0.$$

Prove that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$, which converges μ-almost everywhere to f. Must there be: $\lim_{n\to\infty} \int_X |f_n - f|\ \mathrm{d}\mu = 0$?

Choose a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which satisfies:

$$\mu\left\{x \in X,\ |f_{n_k}(x) - f(x)| > \frac{1}{k}\right\} \leq \frac{1}{k^2}.$$

Consider the sets $A_k = \{x \in X,\ |f_{n_k}(x) - f(x)| > 1/k\}$ and define:

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

We have:

$$\mu(A) = \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(A_k) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{k^2} = 0.$$

Now, if

$$x \in X \setminus A = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} (X \setminus A_k),$$

then

$$\|f_{n_k}(x) - f(x)\| \leq 1/k, \qquad \forall k \geq n, \qquad \textit{for some large } n,$$

and thus $f_{n_k}(x)$ converges to $f(x)$.

Or course it does not have to be $\int_X |f_n - f|\ d\mu \to 0$, because take $X = \mathbf{R}$, $f = 0$ and $f_n = n \cdot \chi_{[0,1/n]}$.

Problem 3. (25 points)
Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be (Lebesgue) measurable.

(i) Prove that the function $g : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}$ given by:

$$g(x,t) = \chi_{\{x;\ |f(x)|>t\}}(x) \quad \forall x \in \mathbf{R}^n\ \forall t \in \mathbf{R}$$

is measurable. (χ stands for the characteristic function of the set given in the subscript).

(ii) Prove that for every $p \geq 1$ one has:

$$\int_{\mathbf{R}^n} |f(x)|^p \ \mathrm{d}x = p \cdot \int_{[0,\infty)} t^{p-1} \cdot \mu\{x \in \mathbf{R}^n;\ |f(x)| > t\}\ \mathrm{d}t$$

(of course, both integrals are the Lebesgue integrals, and μ is the Lebesgue measure on $\mathbf{R}^n$).

(i) The function g is measurable iff the following set is measurable:

$$\{(x,t);\ |f(x)| > t\} = \{(x,t);\ |f(x)| - t > 0\}.$$

Thus it is enough to check that the function $F : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}$, $F(x,t) = |f(x)| - t$ is measurable. This is so, because both functions $(x,t) \mapsto t$ and $(x,t) \mapsto |f(x)|$ are measurable. (see pb 1 (i) for an explanation).

(ii) We have:

$$p \cdot \int_{[0,\infty)} t^{p-1} \cdot \mu\{x \in \mathbf{R}^n;\ |f(x)| > t\}\ dt = p \cdot \int_{[0,\infty)} t^{p-1} \cdot \int_{\mathbf{R}^n} g(x,t)\ dx\ dt.$$

Now use Fubini's theorem to the nonnegative measurable function $(x,t) \mapsto t^{p-1} g(x,t)$. The last integral is equal to:

$$\int_{\mathbf{R}^n} \int_{[0,\infty)} p t^{p-1} g(x,t)\ dt\ dx.$$

Now notice that for a fixed x, $g(x,t) = \chi_{[0,|f(x)|)}$, so the integral above equals to:

$$\int_{\mathbf{R}^n} \int_{[0,|f(x)|]} p t^{p-1}\ dt\ dx = \int_{\mathbf{R}^n} |f(x)|^p\ dx,$$