MIDTERM 4, Math 8602, Spring 2007

Solve problem 1, and ONE of the problems 2 and 3. Clearly indicate your choice. Good luck!

Problem 1. (40 points: 5 points each, 1 point for correct answer, 4 points for correct explanation) Let E be a Banach space. True or False?

(i) The kernel of any operator $T \in \mathcal{L}(E, E)$ is weakly closed.
True: use that a closed convex set is weakly closed.

(ii) If $E^{**} = E^{****}$ (the equality given by the usual map J), then E is reflexive.
True: use that E is reflexive iff E^* is reflexive.

(iii) In l_1 weak topology = strong topology.
False: l_1 is infintely dimensional.

(iv) In the dual of a reflexive Banach space weak convergence = strong convergence.
False: l_2 and take the Schauder basis sequence.

(v) In a reflexive Banach space, a convex, closed and bounded set is weakly compact.
True: closed + convex implies weakly closed. Boundedness therefore additionally implies weakly compact by Kakutani's theorem.

(vi) In c_0 every convex, closed and bounded set is weakly compact.
False: c_0 is not reflexive so think of a unit closed ball and use Kakutani's theorem.

(vii) The function $E \ni x \mapsto \|x\| \in \mathbf{R}$ is continuous from weak to weak topology.
False: the sequence given by the Schauder basis in l_2 converges weakly to 0.

(viii) For any subet $A \subset E$, the set $A^{\perp}$ of functionals annihilating A is weak* closed in E^*.
True: $A^{\perp} = \bigcap_{x \in A} (Jx)^{-1}(0)$ and each $(Jx)^{-1}(0)$ is weak closed as each Jx is weak* continuous on E^*.*

Problem 2. (25 points)
Let E be a uniformly convex Banach space. Let $x_n \rightharpoonup x$ in E and

$$\limsup ||x_n|| \leq ||x||.$$

Prove that x_n converges strongly to x.

[Hint: When $x \neq 0$, consider the sequence $y_n = x_n/(\max\{||x_n||, ||x||\})$, which converges weakly to y.]

First of all, observe that by the weak lowersemicontinuity of the limit, we obtain $||x_n|| \to ||x||$. So if $x = 0$, there is nothing to prove. If $x \neq 0$, follow the hint. Clearly y_n converges weakly to $y = x/||x||$. Therefore, $(y_n + y)/2$ converges weakly to y and by using the weak lowersemicontinuity of the norm again we obtain:

$$1 = ||y|| \leq \liminf ||(y_n + y)/2|| \leq \limsup ||(y_n + y)/2|| \leq 1.$$

Hence the numeric sequence $||(y_n + y)/2||$ converges to 1. By uniform convexity, it implies $||y_n - y|| \to 0$. This concludes the proof.

Problem 3. (25 points)
Let Ω be a ball in $\mathbf{R}^n$. Let S be a linear subspace of $\mathcal{C}(\bar{\Omega})$, that is closed as a subspace of $L^2(\Omega)$. Prove that:

(i) There exists $C > 0$ such that for every $f \in S$: $\|f\|_{L^\infty} \leq C\|f\|_{L^2}$.
(ii) For each $x \in \bar{\Omega}$ there exists a function $g_x \in L^2(\Omega)$ such that $f(x) = \int_\Omega g_x f$ for all $f \in S$.
(iii) S must be finitely dimensional.
[Hint: For example, you may prove that the closed unit ball in S is compact.]

Notice that $Id : \mathcal{C}(\bar{\Omega}) \longrightarrow L^2(\Omega)$ *is continuous, as* $\|f\|_{L^2} \leq \mu(\Omega)^{1/2}\|f\|_{L^\infty}$*. Thus* S *is a closed subspace of* $\mathcal{C}(\bar{\Omega})$*, and* $(S, \|\cdot\|_{L^2})$ *and* $(S, \|\cdot\|_{L^\infty})$ *are Banach spaces with equivalent norms (by the closed graph theorem). This proves (i).*

For every $x \in \bar{\Omega}$*,* $f \mapsto f(x)$ *is a linear continuous functional on* $(S, \|\cdot\|_{L^\infty})$ *and therefore also on* $(S, \|\cdot\|_{L^2})$*. By Hahn-Banach Theorem it may be extended to a linear continuous functional on the whole* $L^2(\Omega)$ *which implies (ii).*

To prove (iii), take a bounded sequence f_n *in* $(S, \|\cdot\|_{L^2})$*. By reflexivity of* L^2*, it must have a weakly converging subsequence (to* $f \in S$*). In particular, since evaluations are linear continuous functionals on* S*,* f_{n_k} *converges to* f *pointwise. But* $\|f_n\|_{L^\infty}$ *is bounded, so* f_{n_k} *must converge to* f *strongly in* L^2 *by the Lebesgue dominated convergence theorem.*