MIDTERM 5, Math 8602, Spring 2007

Solve all three problems. Good luck!

Problem 1. (20 points)
Let Ω be an open, bounded subset of $\mathbf{R}^n$ and let $u_n,\ u \in L^p(\Omega)$, for some $1 < p < \infty$. Assume that:

$$u_n \longrightarrow u \text{ a.e.} \quad \text{and} \quad \|u_n\|_{L^p(\Omega)} \longrightarrow \|u\|_{L^p(\Omega)}.$$

Prove that u_n converge to u strongly in $L^p(\Omega)$.

[Reminder: Referring to "homework" from, say, 3 weeks ago, is not permitted]

[Hint: You may use Egoroff's theorem]

Proof no 1. *Fix $\epsilon > 0$. We know that there exists $\delta > 0$ so that $\int_A |u|^p < \epsilon$ for every measurable set A of measure $< \delta$. By Egoroff's theorem, there exists A like this, that $u_n \longrightarrow u$ in $L^\infty(\Omega \setminus A)$. Thus:*

$$\|u_n - u\|^p_{L^p(\Omega)} \leq \|u_n - u\|^p_{L^p(\Omega\setminus A)} + \|u_n - u\|^p_{L^p(A)}.$$

The first term above is $< \epsilon$, for n large (by the uniform convergence). The second one is estimated by: $C(\int_A |u|^p + \int_A |u_n|^p)$.

Now, notice that $\int_A |u_n|^p < \epsilon$, for n sufficiently large. Because otherwise we would have: $\lim \int_A |u_n|^p \geq \epsilon$, $\int_A |u|^p < \epsilon$, and $\int_{\Omega\setminus A} |u_n|^p \to \int_{\Omega\setminus A} |u|^p$ (by the uniform convergence), which would contradict the convergence of norms on Ω.

Therefore, $\|u_n - u\|^p_{L^p(\Omega)} \leq \epsilon + 2C\epsilon$, for large n, which proves the claim. Note that this proof is also good for $p = 1$.

Proof no 2. *One can also argue like this. $L^p(\Omega)$ is reflexive, so the bounded sequence $\{u_n\}$ must have a weakly converging subsequence $u_{n_k} \rightharpoonup v$ in $L^p(\Omega)$. Now, since u_n converge pointwise a.e. to u, there must be $u = v$ (this was indeed a homework three weeks ago, and I would need to see a proof of this fact in your solution). On the other hand, we know that $L^p(\Omega)$ is uniformly convex (as it is reflexive), so weak convergence and convergence of norms imply strong convergence: $u_{n_k} \longrightarrow u$ in $L^p(\Omega)$. The same reasoning can be made for any subsequence of $\{u_n\}$, hence the whole sequence converges strongly to u. Note, that this proof does not use the boundedness of Ω.*

Problem 2. (20 points)
Let $1 \leq n < p < \infty$. Prove that any function $u \in W^{1,p}(\mathbf{R}^n)$ satisfies:

$$\lim_{\|x\| \to \infty} u(x) = 0.$$

(which means: $\forall \epsilon > 0 \quad \exists R \qquad \|x\| > R \implies |u(x)| < \epsilon$.)

First, by Friedrich's lemma, we have $u_n \to u$ in $W^{1,p}(\mathbf{R}^n)$, for some sequence of functions $u_n \in \mathcal{C}_c^\infty(\mathbf{R}^n)$. Use Morrey's embedding theorem to get:

$$\|u_n - u\|_{L^\infty} \leq C\|u_n - u\|_{W^{1,p}} \to 0.$$

Thus u is the uniform limit of functions in $\mathcal{C}_c^\infty$ and we are done, because: for a given $\epsilon > 0$, there is $\|u_n - u\|_{L^\infty} < \epsilon$, for some large n; take R so big that supp $u_n \subset B(0, R)$.

Problem 3. (20 points)
Given an open set $\Omega \subset \mathbf{R}^n$ and $\frac{n}{n-1} \leq p < \infty$, prove the following inequality:

$$\forall u \in W_0^{1,p}(\Omega) \qquad \|u\|_{L^p(\Omega} \leq S\left(\frac{pn}{p+n}, n\right) \cdot \mu(\Omega)^{1/n} \cdot \|\nabla u\|_{L^p(\Omega)},$$

where $S(q, n)$ stands for the constant in the Sobolev-Gagliardo-Nirenberg embedding $W^{1,q}(\mathbf{R}^n) \longrightarrow L^{q^*}(\mathbf{R}^n)$.

Let $q = \frac{pn}{p+n}$. Verify that $1 \leq q < n$ and that $q^* = p$. Then, by the Sobolev-Gagliardo-Nirenberg inequality, and Holder's inequality, we have:

$$\|u\|_{L^p(\mathbf{R}^n)} \leq S(q, n)\|\nabla u\|_{L^q(\mathbf{R}^n)} \leq S(q, n)\mu(\Omega)^{1/n}\|\nabla u\|_{L^p(\mathbf{R}^n)},$$

which gives the result, because $u \in W_0^{1,p}(\Omega)$ may be extended to a $W^{1,p}(\mathbf{R}^n)$ function by putting 0 outside of Ω (and the norms are clearly the same).