8601 & 8602 Real Analysis Lecture Notes 2006-2007 Prof. Marta Lewicka

Walter Rusin
School of Mathematics
University of Minnesota

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Preface

This is a collection of lecture notes taken at a two-semester graduate-level course in Real Analysis at the School of Mathematics of the University of Minnesota given by Prof. Marta Lewicka during the academic year 2006-2007.

The notes are still under construction. If you spot an error or think that a paragraph might need some reconsideration please mail me.

Contents

Chapter 1

Functional analysis and main functional spaces

1.1 Banach-Steinhaus theorems

Theorem 1.1.1 (Banach-Steinhaus) *Let E be a Banach space and F a normed space. Let $\{T_\alpha\}_{\alpha\in A}$ be a family of operators in $\mathcal{L}(E,F)$. Assume that for all $x \in E$ $\{T_\alpha(x)\}$ is bounded in F. Then $\sup_{\alpha\in A}\|T_\alpha\|_{\mathcal{L}(E,F)} < \infty$.*

<u>Proof</u>: Define sets $X_n = \{x \in E; \forall\alpha \in A \|T_\alpha(x)\| \leq n\}$. Each X_n is closed as an intersection of $T_\alpha^{-1}(cl(B_F(0,n)))$ over all $\alpha \in A$. However $E = \bigcup_n X_n$. Hence by Bair's category theorem at least one of the sets X_n has non-empty interior, say X_{n_0}. So $B(x_0, r) \subset X_{n_0}$ for some $x_0 \in E$ and $r > 0$. For every $x \in B(x_0, r)$ we have $\|T_\alpha(x)\| \leq n$. From this follows that for every $\|x\| \leq 1$ and $\alpha \in A$ $\|T_\alpha(x)\| = \|T_\alpha(\frac{1}{r}(rx+x_0-x_0))\| \leq \frac{1}{r}\|T_\alpha(rx+x_0)\| + \frac{1}{r}\|T_\alpha(x_0)\| \leq \frac{1}{r}n + \frac{1}{r}\|T_\alpha(x_0)\|$. □

Corollary 1.1.2 *Let $\{T_n\}$ be a sequence of operators in $\mathcal{L}(E,F)$ converging pointwise. Then $T_n(x) \to f(x)$ and*

(i) $\sup_n \|T_n\| < \infty$

(ii) $f(x) = T(x)$ *and* $T \in \mathcal{L}(E,F)$

(iii) $\|T\|_{\mathcal{L}(E,F)} \leq \liminf_n \|T_n\|_{\mathcal{L}(E,F)}$.

<u>Proof</u>: (i) For all x $\{T_n(x)\}$ is bounded hence by the Banach-Steinhaus theorem we have $\sup_n \|T_n\| < \infty$.

(ii) Define $f(x) = T(x)$. Linearity follows from properties of limits. Let $\|x\| \leq 1$. For every n we have $\|T_n(x)\| \leq \|T_n\| \cdot \|x\|$. Notice that $\|T\| \leq \sup_n \|T_n\|$.

(iii) Notice that $\|T\| \leq \sup_{n \geq N} \|T_n\|$ and take a sequence of operators T_n converging to $\liminf_n \|T_n\|$. Since the left hand-side does not depend on N we obtain the claim. □

Corollary 1.1.3 *Let E be just a normed space. Let $A \subset E$ be such that $\forall T \in E^*$ $T(A)$ is bounded. Then A is bounded.*

PROOF: Use the Banach-Steinhaus theorem to E^* and the set $\{J(x)\}_{x \in A} \subset \mathcal{L}(E^*, \mathbb{R}) = E^{**}$. For every S We have $|J(x)(S)| = |S(x)|$ which is bounded for every x. So $J(x)$ is bounded but J is an isometry (see proceeding sections for the exact definition of J). □

1.2 Banach theorem of open mapping, closed graph

Theorem 1.2.1 *(Open mapping theorem) Let E, F be two Banach spaces. If $T : E \to F$ is linear, continuous and surjective, then T is open, i.e. for an open set $U \subset E$ the set $T(U)$ is open in F.*

PROOF: First notice that it is enough to prove that there exists an $r > 0$ such that $T(B(0,1)) \supset B(0,r)$. We will show that $cl(T(B(0,1))) \supset B(0,r)$. Define the sets $A_n = cl(T(B(0,n)))$. Notice that $\bigcup_n A_n = F$. By the Bair's category theorem some of the A_n has a non-empty interior. But since $A_n = n \cdot cl(T(B(0,1)))$ we obtain that $cl(T(B(0,1))) \supset B(y_0, 2r)$ for some $y_0 \in F$ and $r > 0$. Since $cl(T(B(0,1)))$ is symmetric it also contains the point $-y_0$. So $B(0,2r) \subset cl(T(B(0,1))) + cl(T(B(0,1)))$. By convexity $cl(T(B(0,1))) + cl(T(B(0,1))) = 2cl(T(B(0,1)))$. Using a simple scaling argument we have $B(0,r) \subset cl(T(B(0,1)))$.

We know now that $cl(T(B(0,1))) \supset B(0,2l)$ for some $l > 0$. We will show that $T(B(0,1)) \supset B(0,l)$. Fix $y \in F$, $\|y\| < l$. We want to show that $y = T(x_0)$ for some $\|x_0\| < 1$. For every $\epsilon > 0$ there is a $z \in E$ with $\|z\| < \frac{1}{2}$ such that $\|y - Tz\| < \epsilon$. In particular take $\epsilon = \frac{l}{2}$. There exists z_1 with $\|z_1\| < \frac{1}{2}$ such that $\|y - Tz_1\| < \frac{l}{2}$. There exists z_2 with $\|z_2\| < \frac{1}{4}$ such that $\|y - Tz_1 - Tz_2\| < \frac{l}{4}$, for $\|4(y - Tz_1) - Tw\| < r$ for some w, $\|w\| < 1$ and set $z_2 = \frac{w}{4}$. Continue this procedure. There exists a z_n with $\|z_n\| < \frac{1}{2^n}$ such that $\|y - T(z_1 + z_2 + \ldots + z_n)\| < \frac{l}{2^n}$. Let $x_n = z_1 + z_2 + \ldots + z_n$. We have $\|y - T(x_n)\| \to 0$ as $n \to \infty$ which means $T(x_n) \to y$ and by continuity of T we have $T(x_n)\ to T(x_0)$ where $x_n \to x_0$. Thus $T(x_0) = y$. □

There are three major classical consequences of this theorem.

Corollary 1.2.2 *Let E, F be two Banach spaces and let $T \in \mathcal{L}(E, F)$. If T is invertible then $T^{-1} \in \mathcal{L}(F, E)$.*

Corollary 1.2.3 *If $(E, \|\cdot\|)$ and $(E, \|\cdot\|_1)$ are Banach spaces and there is a constant $C > 0$ such that for all $x \in E$ we have $\|x\| \leq C\|x\|_1$, then $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.*

Corollary 1.2.4 *(Closed Graph theorem) If $T : E \to F$ is a linear map between Banach spaces E, F then $T \in \mathcal{L}(E, F)$ if and only if $Graph(T) = \{(x, Tx); x \in E\}$ is closed in $E \times F$.*

1.3 Weak topologies and weak convergence, weak-$*$ topologies and weak-$*$ convergence, Mazur's theorem, the separation theorems (geometrical form of Hahn-Banach)

Let E be a normed space (does not have to be Banach). We can also consider E^* - the dual of E. We can define the second dual $E^{**} = (E^*)^*$. We say that $\phi \in E^{**}$ if $\phi \in \mathcal{L}(E^*, \mathbb{R})$. There is a particular class of elements of E^{**}.

There exists a natural ebedding $J : E \to E^{**}$ defined as $J(x)(T) = T(x)$. The mapping J turns out to be linear and continuous. What is its norm? We have $|J(x)(T)| = |T(x)| \leq \|T\| \cdot \|x\|$ so $\|J(x)\| \leq \|x\|$. By the Hahn-Banach theorem we have $\|J(x)\| = \|x\|$. Hence J is an isometry onto its image.

Definition 1.3.1 *We call a space E reflexive if $J(E) = E^{**}$.*

Of course, not all spaces are reflexive. Consider c_0. We know that $(c_0)^* = \ell_1$, $(\ell_1)^* = \ell^\infty$. So $c_0^{**} = \ell^\infty$. However every finitely dimensional space $\mathbb{R}^n$ is reflexive.

Definition 1.3.2 *We say that $x_n \rightharpoonup x$ in E (converges weakly) if and only if for every $T \in E^*$ we have $T(x_n) \to T(x)$. We say that $T_n \overset{*}{\rightharpoonup} T$ in E^* (converges weakly-$*$) if and only if for every $x \in E$ we have $T_n(x) \to T(x)$.*

Notice that the weak-$*$ convergence is essentially the pointwise convergence. By the above definition we can equip E^* with three types of convergence:

- strong $T_n \to T$ if $\|T_n - T\| \to 0$

- weak $T_n \rightharpoonup T$ if for all $\phi \in E^{**}$ we have $\phi(T_n) \to \phi(T)$
- weak-$*$ $T_n \overset{*}{\rightharpoonup} T$ if for all $x \in E$ we have $T_n(x) \to T(x)$.

Lemma 1.3.3 *In any normed space we have:*
(i) strong convergence implies weak convergence
(ii) strong convergence implies weak-$*$ convergence
(iii) weak convergence implies weak-$*$ convergence
(iv) no converse implication in general is true.

<u>PROOF</u>: (i) If $x_n \to x$ then $|T(x_n) - T(x)| \leq \|T\| \cdot \|x_n - x\| \to 0$ so $T(x_n) \to T(x)$. This is true for every T.

(ii) Let $T_n \to T$ in E^*. For all $x \in E$ we have $|T_n(x) - T(x)| \leq \|T_n - T\| \cdot \|x\| \to 0$.

(iii) Notice that $T_n \overset{*}{\rightharpoonup} T$ is a special case of weak convergence since $J(x) \in E^{**}$. Notice that the weak convergence is the weak-$*$ convergence if E is reflexive.

(iv) Consider the sequence $e_n \in c_0$. We claim that $e_n \rightharpoonup 0$. However clearly $\|e_n\| = 1$, so no strong convergence is possible. Let $T \in c_0^*$. Then T can be identified with $\{y_1, y_2, \ldots\}$ in ℓ^1. We have $T(e_n) = y_n \to 0$.

Take now c_0^* and take T_n such that $T_n(x) = x_n$ for $x \in c_0$. We claim that $T_n \overset{*}{\rightharpoonup} 0$ but T_n does not converge strongly. Indeed, for all $x \in E$ we have $T_n(x) = x_n \to 0$, however $\|T_n\| = 1$. Also T_n does not converge weakly to 0, for take $\phi \in (\ell^1)^*$ such that $\phi(\{y_n\}) = \sum_{n=1}^{\infty} y_n$. Then $\phi(T_n) = 1$. □

What are the properties of weak and weak-$*$ limits?

Lemma 1.3.4 *The weak and weak-$*$ limits have the following properties*
(i) the weak and weak-$*$ limits are unique
(ii) if $x_n \rightharpoonup x$ then $\|x\| \leq \liminf \|x_n\|$
(iii) if E is a Banach space and $T_n \overset{*}{\rightharpoonup} T$ then $\|T\| \leq \liminf \|T_n\|$.

<u>PROOF</u>: (i) Let $x_n \rightharpoonup x$, $x_n \rightharpoonup y$. For every $T \in E^*$ we have $T(x_n) \to T(x)$ and $T(x_n) \to T(y)$, so $T(x) = T(y)$. By the Hahn-Banach theorem we have $x = y$. Assume now $T_n \overset{*}{\rightharpoonup} T$ and $T_n \overset{*}{\rightharpoonup} S$. For every $x \in E$ we have $T(x) = S(x)$ so $T = S$.

(iii) Has already been proved, follows from the Banach-Steinhaus theorem.

(ii) Let $x_n \rightharpoonup x$ in E. By definition this means that $J(x_n) \overset{*}{\rightharpoonup} J(x)$ in E^{**}. By (iii) we obtain $\|J(x)\| \leq \liminf \|J(x_n)\|$. Since J is an isometry between Banach spaces we are done. □

All three types of convergence come from different topologies on E and E^*.

Definition 1.3.5 *Let E be a normed space. The weak topology on E is the smallest*[1] *topology such that each $T \in E^*$ is continuous. The weak-$*$ topology on E^* is the smallest topology such that every $J(x)$ is continuous.*

Let $\mathcal{O}$ be the smallest topology on E with respect to which every $T \in E^*$ is still continuous. Notice that $\forall T \in E^*$ and $\epsilon > 0$ we have $T^{-1}((-\epsilon, \epsilon)) \in \mathcal{O}$. Furthermore for any finite collection $T_1, T_2, \ldots, T_n \in E^*$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n > 0$ we have

$$\bigcap_{i=1}^{n} T_i^{-1}((-\epsilon_i, \epsilon_i)) = \{x \in E; |T_i(x)| < \epsilon_i \quad \forall_i\} \in \mathcal{O}.$$

Also each set $x_0 + U$, where U is as above, is also an element of $\mathcal{O}$. Furthermore, arbitrary unions of sets as above are also in $\mathcal{O}$. This is a natural property of topology. In fact the weak topology is composed of sets as above. Moreover - sets being translations of $\bigcap_{i=1}^{n} T_i^{-1}((-\epsilon_i, \epsilon_i))$ form a basis of neighborhoods of a point.

We can construct the weak-$*$ topology in a similar fashion.

Theorem 1.3.6 *Let E be a normed space. Then*

(i) the sequence x_n converges to x in the weak topology if and only if $x_n \rightharpoonup x$

(ii) the sequence $T_n \in E^*$ converges to T in the weak-$*$ topology if and only if $T_n \overset{*}{\rightharpoonup} T$

<u>Proof</u>: (i) ($\Rightarrow$) Take $T \in E^*$. T is continuous in the weak topology, and $x_n \to x$ in the weak topology. Since T must be sequentially continuous we have $T(x_n) \to T(x)$. ($\Leftarrow$) Let $x_n \rightharpoonup x$. Take $x \in U \in \mathcal{O}$. Then U must contain a basis set $V = \bigcap_{i=1}^{n} T_i^{-1}((-\epsilon_i, \epsilon_i))$ for some $T_1, T_2, \ldots, T_n \in E^*$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n > 0$. Since $T_i(x_n) \to T_i(x)$ for all $i = 1, \ldots, n$, there is an $M > 0$ such that $\forall n > M$ $|T_i(x_n) - T_i(x)| < \min \epsilon_i$ so for all $n > M$ we have $x_n \in V$ which means $x_n \to x$ in the weak topology.

(ii) by analogy. □

Lemma 1.3.7 *The weak and weak-$*$ topologies are Hausdorff.*

[1] We say "smallest" in the sense that it should contain as few open sets as possible.

PROOF: Take $x \neq y \in E$. By Hahn-Banach theorem there exists $T \in E^*$ such that $T(x) < \alpha < T(y)$. Let $U = T^{-1}(-\infty, \alpha)$ and $V = T^{-1}(\alpha, \infty)$. Then $x \in U$, $y \in V$. Notice that $U \cap V = \emptyset$.

Assume that $T \neq S \in E^*$. There exists an $x \in E$ such that $T(x) \neq S(x)$. Assume $T(x) < \alpha < S(x)$. Let $U = J(x)^{-1}(-\infty, \alpha)$ and $V = J(x)^{-1}(\alpha, \infty)$. We have $T \in U$, $S \in V$ and $U \cap V = \emptyset$. □

Remark 1.3.8 In a finitely dimensional space all three topologies coincide.

Example 1.3.9 Consider the set $S = \{x \in E; \|x\| = 1\}$. If E is infinitely dimensional then S is never closed in the weak topology. However the closure of S in the weak topology is the whole ball $\bar{B} = \{x \in E; \|x\| \leq 1\}$.

First notice that $cl(S) \subset cl(B)$ because $cl(B)$ is weakly closed as it is strongly closed. Take $x_0 \notin cl(B)$. By the Hahn-Banach theorem there is a $T \in E^*$ such that $\|T\| = 1$ and $Tx_0 = \|x_0\| > 1$. However $\forall\ x \in cl(B)$ $|T(x)| \leq 1$ so the set $T^{-1}((1, +\infty))$ contains x_0, is open in the weak topology and is disjoint from $cl(B)$. To show the reverse inclusion, $cl(S) \supset cl(B)$ take $x_0 \in cl(B)$. Let $x_0 \in U \in \mathcal{O}$. Indeed we have $S \cap U \neq \emptyset$. Certainly we also have $U \supset x_0 + \cap_{i=1}^{n} T_i^{-1}((-\epsilon_i, \epsilon_i))$ for some $T_1, T_2, \ldots, T_n \in E^*$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n > 0$. Since E is infinitely dimensional there exists $y_0 \in E \setminus \{0\}$ such that $\forall_i\ T_i(y_0) = 0$ (otherwise $x \mapsto (T_1(x), T_2(x), \ldots, T_n(x))$ would be an isomorphism). Notice that $x_0 + \mathbb{R}y_0 \subset U$. Consider the function $t \mapsto \|x_0 + ty\|$. This map is continuous hence there exists a t such that $\|x_0 + ty\| = 1$.

If E is a normed infinitely dimensional space then every bounded subset of E cannot be weakly open. In particular it has an empty interior with respect to the weak topology.

Corollary 1.3.10 *The weak topology on an infinitely dimensional space is not metrizable*

PROOF: Let E be a normed space. Assume that there is a metric d on E, such that (E, d) gives the weak topology. Consider $B_n = B(0, \frac{1}{n})$ in (E, d). Each B_n is weakly open, so it has to be unbounded in $(E, \|\cdot\|)$. Thus $\forall n \exists x_n \in B_n\ \|x_n\| \geq n$. Notice that $x_n \to 0$ in (E, d) so $x_n \rightharpoonup 0$, but $\{x_n\}$ is unbounded in $(E, \|\cdot\|)$ which gives a contradiction. □

In general weak closedness implies strong closedness but not vice-versa.

Theorem 1.3.11 *In a normed space, a convex set is weakly closed if and only if it is strongly closed.*

PROOF: Notice first that weak closedness implies strong closedness as it is true for every set. Let $A \subset E$ be closed and convex. Assume that there exist a $T \in E^*$ and $\alpha > 0$ such that for every $x \in A$ $Tx < \alpha < Tx_0$. Take $U = T^{-1}(\alpha, +\infty)$. We have $x_0 \in U$, U is weakly open and $U \subset E \setminus A$. $\square$

So, why does such a functional exist? This is essentially the core of the First Separation theorem. Before we prove it, we notice an interesting corollary following from the previous theorem.

Corollary 1.3.12 *(Mazur's theorem) Let $x_n \rightharpoonup x$ in a normed space. Then there exists a sequence of convex combinations of x_n such that it converges strongly to x.*

PROOF: Take the set $A = cl(conv\{x_1, x_2, \ldots\})$ where the closure is taken in the strong topology. A is a strongly closed and convex set hence it is also weakly closed. Since $x_n \rightharpoonup x$ and $x_n \in A$ we have $x \in A$ thus it can be approximated by elements of A which are finite convex combinations of elements of $\{x_n\}$. $\square$

Theorem 1.3.13 *(First Separation theorem) Let E be a normed space, $A \subset E$ a convex and closed or open set, $x_0 \in E \setminus A$. There exists a $T \in E^*$ such that for all $x \in A$ $T(x) < T(x_0)$ in the case when A is open. If A is closed there exists a $T \in E^*$ and $\alpha \in \mathbb{R}$ such that for all $x \in A$ we have $T(x) < \alpha < T(x_0)$.*

To prove the theorem we will need the following lemma.

Lemma 1.3.14 *Let A be an open and convex set with $0 \in A$. Define $p : E \to \mathbb{R}$ as $p(x) = \inf\{\alpha > 0; \frac{x}{\alpha} \in A\}$. Then*
(i) p is a Banach functional
(ii) there exists an M such that for all $x \in E$ $0 \leq p(x) \leq M\|x\|$
(iii) $A = \{x \in E; p(x) < 1\}$.

PROOF: (of lemmma) (ii) Since A is open and $0 \in A$ there is an $r > 0$ such that $B(0, r) \subset A$. Thus $\frac{x}{\|x\|} \cdot \frac{r}{2} \in A$ for any $x \in E$. We thus have $0 \leq p(x) \leq \|x\| \cdot frac2r$. This shows that p is well-defined.

(iii) Let $x \in A$. Then also $(1 + \epsilon)x \in A$ for some $\epsilon > 0$. Thus $p(x) \leq \frac{1}{1+\epsilon} < 1$. If $p(x) < 1$ then there exists $\alpha \in (0, 1)$ such that $\frac{x}{\alpha} \in A$. Notice that $x = \alpha \cdot \frac{1}{\alpha}x + (1 - \alpha) \cdot 0 \in A$ by convexity.

(i) For all $\lambda \geq 0$ and all $x \in E$ we have $p(\lambda x) = \lambda p(x)$. Also, fix some $\epsilon > 0$ so that $\frac{x}{p(x)+\epsilon}, \frac{y}{p(y)+\epsilon} \in A$. We then have

$$\frac{x+y}{p(x)+p(y)+2\epsilon} = \frac{p(x)+\epsilon}{p(x)+p(y)+2\epsilon} \cdot \frac{x}{p(x)+\epsilon} + \frac{p(y)+\epsilon}{p(x)+p(y)+2\epsilon} \cdot \frac{y}{p(y)+\epsilon}.$$

Thus $\frac{x+y}{p(x)+p(y)+2\epsilon} \in A$ and by (iii) we have $p(x+y) \leq p(x)+p(y)+2\epsilon$. Since ϵ was chosen arbitrarily we get the claim. □

PROOF: (of the First Separation theorem) Let E be a normed space.

Case 1: $A \subset E$, convex, open and $x_0 \notin A$. We want to show that there exists a $T \in E^*$ such that for all $x \in A$ $T(x) < T(x_0)$. Without loss of generality we can assume that $0 \in A$. Let $E_0 = \mathbb{R}x_0$. Define $T_0 : E_0 \to \mathbb{R}$, $T_0(tx_0) = tp(x_0)$. Notice that $T_0(x) \leq p(x)$. So by the Hahn-Banach theorem there exists an extension T of T_0 such that $T(x) \leq p(x)$. We claim that $T \in E^*$. Indeed, this is true because $T(x) \leq p(x) \leq M\|x\|$ and $-T(x) \leq p(-x) \leq M\|x\|$ thus $|T(x)| \leq M\|x\|$. Furthermore for every $x \in A$ we have $T(x) \leq p(x) < 1 \leq p(x_0) = T_0(x_0) = T(x_0)$ hence $T(x) < T(x_0)$ for $x \in A$.

Case 2: Let A be convex and closed. Let $x_0 \notin A$. We want to show that there exists $S \in E^*$ and $\alpha, \beta \in \mathbb{R}$ such that for all $x \in A$ $S(x) \leq \alpha < \beta \leq S(x_0)$. Define $A_\epsilon = A + B(0, \epsilon)$. Notice that A_ϵ is open and still convex. Notice that if ϵ is small enough then $x_0 \notin A_\epsilon$. Otherwise we would have $x_0 = a_n + w_n$ where $a_n \in A$, $w_n \in B(0, 1/n)$ so $a_n = x_0 - w_n$ and a_n converges to an element of A as A is closed. The right hand-side converges to $x_0 \notin A$. Now, we use case 1 to get $S \in E^*$ which separates $A_{1/n}$ and x_0. Thus for every $x \in A_{1/n}$ we have $S(x) < 1 \leq S(x_0)$. Notice that there exists a $y_0 \in B(0, \epsilon)$ such that $S(y_0) > 0$. Now, for every $x \in A$ $S(x) = S(x + y_0) - S(y_0) < 1 - S(y_0) < 1 \leq S(x_0)$. □

In E we can distinguish between two topologies: $\sigma(E)$-strong topology, $\sigma(E, E^*)$-weak topology. The dual E^* may be endowed with the following topologies: $\sigma(E^*)$ - strong topology, $\sigma(E^*, E^{**})$-weak topology, $\sigma(E^*, E)$-weak-$*$ topology. In general the topologies are different.

Example 1.3.15 In E^*, take the closed hyperplane $\phi^{-1}(0)$ for some $\phi in E^{**}$. This is a strongly closed, convex set hence it is also weakly closed in E^*. However $\phi^{-1}(0)$ is weak-$*$ closed if and only if $\phi \in J(E)$. So if E is not reflexive then this set may not be weak-$*$ closed.

1.4 Compactness and sequential compactness, the theorems of Banach-Alaoglu, Kakutani, Goldstine

For many application some compactness results are very useful. We know that in an infinitely dimensional space a closed unit ball is not compact with respect to the strong topology. However we may ask if the unit closed ball in E is weakly compact or if E is a dual of some space if the unit closed ball is weakly-$*$ compact?

Theorem 1.4.1 (Banach-Alaoglu) *The unit closed ball in E^* is weakly-$*$ compact.*

PROOF: Consider the space $\mathbb{R}^E = \Pi_{x\in E}\mathbb{R} = \{g : E \to \mathbb{R}\}$ with the product topology. Consider $\Phi : (E^*, \sigma(E^*, E)) \to \mathbb{R}^E$ given by $\Phi(T) = T$. Notice that Φ is continuous, open and $1-1$. So Φ is a homeomorphism between E^* and $\Phi(E^*)$. Now, look at $\Phi(cl(B_{E^*}))$. We have $\Phi(cl(B_{E^*})) = \Phi(E^*) \cap \prod_{x\in E}[-\|x\|, \|x\|]$. This is because $T \in cl(B_{E^*})$ if and only if $|T(x)| \leq \|x\|$. The goal to show that $\Phi(E^*)$ is closed since by Tichonov's theorem $\prod_{x\in E}[-\|x\|, \|x\|]$ is compact. Notice that $g \in \Phi(E^*)$ if and only if for all $x, y \in E$ we have $g(x+y) - g(x) - g(y) = 0$ and if for all $x \in E$ and $\lambda \in \mathbb{R}$ $g(\lambda x) - \lambda g(x) = 0$. Fix $x, y \in E$ and look at the set $\{g \in \mathbb{R}^E : g(x+y) - g(x) - g(y) = 0\}$. This is a closed set because the topology is such that evaluations are continuous. Since this is true for any $x, y \in E$ so $\bigcap_{x,y}\{g \in \mathbb{R}^E : g(x+y) - g(x) - g(y) = 0\}$ is also closed. Similarly $\bigcap_{x\in E,\lambda\in\mathbb{R}}\{g \in \mathbb{R}^E : g(\lambda x) - \lambda g(x)\}$. Taking intersection of these two sets we obtain the claim. $\square$

Theorem 1.4.2 (Kakutani) *Let E be a Banach space. $cl(B_E)$ is weakly compact if and only if E is reflexive.*

PROOF: If E is reflexive then $J(cl(B_E)) = cl(B_{E^{**}})$. By the Banach-Alaoglu theorem $cl(B_{E^{**}})$ is compact with respect to the weak-$*$ topology on E^{**}. Consider $J^{-1} : (E^{**}, \sigma(E^*, E)) \to (E^{**}, \sigma(E, E^*)$. If J^{-1} is continuous we are done. Take any $T \in E^*$. Consider $(E^{**}, \sigma(E^*, E)) \stackrel{J^{-1}}{\to} (E, \sigma(E, E^*)) \stackrel{T}{\mathbb{R}}$. The composition $T \circ J^{-1}$ is continuous. From this follows continuity of J^{-1}. To prove the reverse implication we need an auxiliary result. $\square$

Theorem 1.4.3 (Goldstine) *Let E be a Banach space. Then $J(cl(B_E))$ is dense in $cl(B_{E^{**}})$ with the weak-$*$ topology.*

Remark 1.4.4 Notice that this image is closed in the strong topology. So it cannot be be dense in the strong topology unless E is reflexive.

<u>PROOF</u>: (ot Goldstine's theorem) Take $\phi \in cl(B_{E^{**}})$ and $U = \{\eta \in E^{**}; |(\eta - \phi)(T_i)| < \epsilon, i = 1, \dots, n\}$ for some $T_1, \dots, T_n \in E^*$. We want to show that there exists an $x_\epsilon \in E$ so that for $i = 1, \dots, n$ $|T_i(x_\epsilon) - \phi(T_i)| < \epsilon$ and $x_\epsilon \in cl(B_E)$. Suppose it is not true. Consider $(T_1, T_2, \dots, T_n) : E \to \mathbb{R}^n$. Look at $\phi(T_1), \phi(T_2), \dots, \phi(T_n)$. At least one of the numbers is bigger then ϵ thus this point is outside of the image of the ball. There exists $(\alpha_1, \dots, \alpha_n) \in R^n$ such that for all $x \in cl(B_E)$ we have $\sum_{i=1}^n \alpha_i T_i(x) < \alpha < \sum_{i=1}^n \alpha_i \phi(T_i)$. Now $\sum_{i=1}^n \phi(T_i) = \phi(\sum_{i=1}^n \alpha_i T_i)$. Since $\|\phi\| \leq 1$ we have $|\sum_{i=1}^n \alpha_i T_i(x)| \leq \|\phi\| \cdot \|\sum_{i=1}^n \alpha_i T_i\|$ and separation is not possible. □

<u>PROOF</u>: (continuation of the proof of Kakutani's theorem) Assume that $cl(B_E)$ is weakly compact. $J : E \to E^{**}$ is strongly continuous. This is equivalent to continuity in weak topologies. Hence $J : (E, \sigma(E, E^*)) \to (E^{**}, \sigma(E^{**}, E)$ is continuous. Thus $J(cl(B_E))$ is weak-$*$ compact. By Goldstine's theorem there must be $J(cl(B_E)) = cl(B_{E^**}$. So also $J(E) = E^{**}$ by scaling. □

Corollary 1.4.5 *Let E be reflexive. Let $A \subset E$ be closed, convex and bounded. Then A is weakly compact.*

<u>PROOF</u>: A is closed and convex thus it must be weakly closed. Becaue it is bounded it is contained in some ball $cl(B(0, r)) = rcl(B_E)$ which is weakly compact. □

Corollary 1.4.6 *A Banach space E is reflexive if and only if E^* is reflexive.*

<u>PROOF</u>: Let E be reflexive. Consider $cl(B_{E^*})$, it is weakly-$*$ compact by the Banach-Alaoglu theorem. But E is reflexive so $cl(B_{E^*})$ is also weakly compact. So E^* is reflexive.

If E^* is reflexive so is E^{**} (by the previous argument). Look at $J(E)$. It is a closed subspace of E^{**}. We use the fact that a closed subspace of a reflexive space is also reflexive and the fact that linear isometry preserves reflexivity. □

Fact 1.4.7 A closed subspace of a reflexive Banach space is also reflexive.

PROOF: Let $F \subset E$ be a closed subspace. $cl(B_F) = F \cap cl(B_E)$. Since $cl(B_E)$ is weakly compact in E we know that $cl(B_F)$ is weakly compact in E. Consider $i : f \to E$. This is a strongly continuous map so it is also continuous between weak topologies and for any open set in E its restriction to F is also open. Since $\sigma(E, E^*)$ is a topology it is enough to check the inclusion on subbasis - sets of the form $T^{-1}(-\epsilon, \epsilon)$, $T \in F^*$. By the Hahn-Banach theoerem there is an $S \in E^*$ such that $S|_F = T$ and we have $S^{-1}(-\epsilon, \epsilon) = T^{-1}(\epsilon, \epsilon)$. □

Theorem 1.4.8 *Let E be a separable Banach space, then $(B_{E^*}, \sigma(E^*, E))$ is metrizable.*

Corollary 1.4.9 *Let E be separable. Then any bounded sequence $T_n \in E^*$ must have a weakly-$*$ converging subsequence.*

PROOF: The proof follows from the Banach-Alaoglu theorem and the previous theorem. Since T_n is bounded, it is contained in some ball. So it is weak-$*$ compact but the weak-$*$ topology is metrizable hence T_n is also sequentially compact. □

1.5 Reflexivity, separability, uniform convexity of Banach spaces

Theorem 1.5.1 *Let E be a Banach space. If E^* is separable then E is also separable.*

[PROOF]

Theorem 1.5.2 *Let E be a Banach space. Then E is reflexive and separable if and only if E^* is reflexive and separable.*

PROOF: We already know that if E^* is reflexive and separable so is E. This follows from previous theorems [PUT NUMBERS]. For the reverse implication notice that E^{**} is reflexive and separable because E is reflexive and separable. Now use the previous implication and get separability and reflexivity of E^*. □

Theorem 1.5.3 *Let E be a reflexive Banach space. Any bounded sequence in E has a weakly converging subsequence.*

PROOF: Define $F = cl(span\{x_n\})$. It is a closed subspace of E. Notice that F is a closed subspace of E hence also a Banach space. If we consider linear combinations of x_n with rational coefficients, it turns out that F is separable. Therefore in F^{**} every bounded sequence has a weak-$*$ converging subsequence. Look at $\{J(x_n)\}$. It has a subsequence $J(\{x_{n_k}\})$ which converges weakly-$*$ in F^{**}. But F is reflexive so from this follows that $x_{n_k} \rightharpoonup x_0$ in F. Thus, since $E^* \subset F^*$ we are done. □

Definition 1.5.4 *A normed space E is called uniformly convex if and only if $\forall \epsilon > 0\ \exists \delta > 0\ \forall x, y \in E\ \|x\|, \|y\| \leq 1$, $\|x - y\| > \epsilon$ implies $\left\|\frac{1}{2}(x + y)\right\| < 1 - \delta$.*

Theorem 1.5.5 *If a Banach space is uniformly convex then it must be reflexive.*

PROOF: Let E be a Banach space. Take $\phi \in E^{**}$. We want to show that $\phi = J(x)$ for some $x \in E$. It is enough to assume that $\|\phi\| = 1$. We will prove that $\forall \epsilon > 0\ \exists x \in cl(B_E)$ such that $\|\phi - J(x)\| \leq \epsilon$. $J(cl(B_E)) \subset E^{**}$ is closed so we can take a sequence approximating ϕ. Take $\epsilon > 0$ and $\delta > 0$ from the uniform convexity. Take some $T \in E^*$, $\|T\| = 1$. Define $U = \{\eta \in E^{**}; |(\eta - \phi)(T)| < \delta/2\}$. It is open in the weak-$*$ topology in E^{**}. By Goldstine's theorem there exists $x \in cl(B_E)$ such that $J(x) \in U \cap J(cl(B_E))$. We show that this is a good x, that is $\|\phi - J(x)\| \leq \epsilon$. If not then $\|\phi - J(x)\| > \epsilon$ and consider $\phi \in E^{**} \setminus cl(B(J(x), \epsilon)) = V$. This set is weakly-$*$ open so there exists $y \in cl(B_E)$ such that $J(y) \in U \cap V$. We have $|T(x) - \phi(T)| < \delta/2$ and $|T(y) - \phi(T)| < \delta/2$. So $|T(x + y) - 2\phi(T)| < \delta$. Since $T(x + y) + \delta \leq \|x + y\| + \delta$ and $T(x + y) + \delta > 2\phi(T) > 2 - \delta$ we also require $\phi(T) > 1 - \delta/2$. Hence if $\|x + y\| \geq 2 - 2\delta$ then $\|x - y\| \leq \epsilon$. But $\|J(x) - J(y)\| > \epsilon$ hence we get a contradiction. □

1.6 Hilbert spaces, Riesz-Frechet theorem, projection theorem, Lax-Milgram theorem, Hilbert sums and Hilbert basis, Bessel-Parseval identity

Definition 1.6.1 *A Hilbert space is a Banach space whose norm is induced by a scalar product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$. We consider $\|x\| = \sqrt{\langle x, x \rangle}$.*

Lemma 1.6.2 (Parallelogram identity) *We have $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.*

PROOF: we have $\langle x+y, x+y\rangle + \langle x-y, x-y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle + \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle = 2\langle x,x\rangle + 2\langle y,y\rangle = 2(\|x\|^2 + \|y\|^2)$. □

Lemma 1.6.3 *Every Hilbert space is uniformly convex.*

PROOF: Take $\epsilon > 0$ and $x, y \in H$ satisfying the conditions in the definition of uniform convexity. We have

$$\left\|\frac{x+y}{2}\right\| = \frac{1}{2}(\|x\|^2 + \|y\|^2) - \left\|\frac{x-y}{2}\right\|^2 \le \frac{1}{2}(1+1) - \frac{\epsilon^2}{4} = 1 - \frac{\epsilon^2}{4}.$$

Take $\delta < 1 - (1-\epsilon^2/4)^{1/2}$. □

Uniform convexity implies strict convexity. If not then there are $x, y \in H$ such that $\|x\| = \|y\| = 1$, but such that for some $t \in (0,1)$ $\|z\| = \|tx + (1-t)y\| = 1$. Then $z = \alpha x + (1-\alpha)\frac{x+y}{2}$ hence $\|z\| \le \alpha\|x\| + (1-\alpha)\left\|\frac{x+y}{2}\right\| < 1$. This is a contradiction.

Theorem 1.6.4 (Riesz-Fréchet) *Let H be a Hilbert space. For every $T \in H^*$ there is a unique $x \in H$ such that $T(y) = \langle x, y\rangle$ for every $y \in H$.*

PROOF: Consider the map $x \mapsto \langle x, \cdot\rangle$. Call this map S. S is certainly linear and $\|S(x)\| = \|x\|$. So S is an isometry. The goal is to show that S is onto. We will show that $S(H)$ is dense in H^*.

If $cl(S(H)) \ne H^*$ then there is a $\phi \in H^{**}$ such that $\phi|_{S(H)} = 0$ but $\phi \ne 0$. However since H is uniformly convex it is also reflexive thus $H^{**} = H$. It means that $\phi(T) = T(x_0)$ for some x_0 and $\langle x, x_0\rangle = 0$ for all $x \in H$. So $\langle x_0, x_0\rangle = 0$ and this implies $x_0 = 0$ which shows $\phi = 0$. This is a contradiction. □

A consequence of this theorem is the following fact. Let $M \subset H$ be some closed subspace. Then $M^\perp \subset H^*$ may be identified with $\{x \in H : \langle x, y\rangle = 0, \forall y \in M\}$. Then $M \cap M^\perp = \{0\}$ and $M \oplus M^\perp = H$.

Theorem 1.6.5 *In a Hilbert space, every closed subspace is complemented.*

Lemma 1.6.6 (Cauchy-Schwarz inequality) *In a Hilbert space for all $x, y \in H$ we have*

$$|\langle x, y\rangle| \le \|x\| \cdot \|y\|.$$

PROOF: Look at the following function $t \mapsto \|x+ty\|^2$. Then $\langle x+ty, x+ty\rangle = \langle x,x\rangle + 2\langle x,y\rangle + t^2\langle y,y\rangle$. This is a quadratic in t and $\langle x,x\rangle + 2t\langle x,y\rangle t^2\langle y,y\rangle \geq 0$. So $\Delta \leq 0$ and $4\langle x,y\rangle^2 - 4\langle x,x\rangle\langle y,y\rangle \leq 0$ which gives the inequality. □

Theorem 1.6.7 *In a Hilbert space every closed subspace is complemented. Moreover, if in a Banach space every closed subspace is complemented then it must be isomorphic to some Hilbert space.*

PROOF: Consider M, a closed subspace of H a Hilbert space. Consider $M^\perp$. In terms of Hilbert space we have $M^\perp = \{x_0 \in H; \langle x_0, x\rangle = 0 \forall x \in M; \langle x_0, x\rangle = 0 \ \forall x \in M\}$. It is easy to see that $M \cap M^\perp = \{0\}$. Why is $M \oplus M^\perp = H$? In other words, why given $x \in H$ there exist unique $x_0 \in M$ and $x_1 \in M^\perp$ such that $x = x_0 + x_1$. We will need another theorem. □

Theorem 1.6.8 *Let $K \subset H$ be convex and closed. Then for every $x \in H$ there exists exactly one element $P_K(x)$ in K such that $\|x - P_K(x)\| = dist(x,K)$. Moreover $P_K(x)$ is characterized by the following property $P_K(x) \in K$ and $\langle x - P_K(x), y - P_K(x)\rangle \leq 0$ for all $y \in K$.*

PROOF: Step 1: (Existence of $P_K(x)$) Let (y_n) be a sequence in K such that $\|y_n - x\| \to dist(x,K)$. We prove that (y_n) is Cauchy. Use the parallelogram identity to $x - y_n$ and $x - y_m$. We have

$$\|2x - (y_n + y_m)\|^2 + \|y_n - y_m\|^2 = 2(\|x - y_n\|^2 + \|x - y_m\|^2),$$

hence

$$\|y_n - y_m\|^2 = 2\left(\|x - y_n\|^2 + \|x - y_m\|^2 - 2\left\|x - \frac{y_n + y_m}{2}\right\|^2\right) \leq$$

$$2\left(\|x - y_n\|^2 + \|x - y_m\|^2 - 2dist(x,K)^2\right).$$

As n, m are getting larger, the right hand-side becomes smaller thus (y_n) is a Cauchy sequence.

Step 2: (Equivalence) Let $z \in K$ such that $\|z - x\| = dist(x,K)$. Consider the segment $[z,y]$. It follows that $\|z - x\|^2 \leq \|tz + (1-t)y - x\|^2 = \|z - x + (1-t)(y-z)\|^2 = \|z - x\|^2 + (1-t)^2\|z - y\|^2 - 2(1-t)\langle x - z, y - z\rangle$. So $2\langle x - z, y - z\rangle \leq (1-t)\|y - z\|^2$. Since this is true for every $t \in [0,1]$ we have $\langle x - z, y - z\rangle \leq 0$. If $z \in K$ and $\langle z - x, z - y\rangle \leq 0$ for all $y \in K$ then we look at $\|z - x\|^2 - \|y - x\|^2 \leq -\|y - z\|^2 - \langle z - x, y - z\rangle \leq 0$.

Step 3: (Uniqueness) Let z_1, z_2 minimize the distance. Then $\langle x - z_1, y - z_1\rangle \leq 0$ and $\langle x - z_2, y - z_2\rangle \leq 0$. So $0 \geq \langle z_2 - z_1, z_2 - z_1\rangle = \|z_1 - z_2\|^2$ and this implies $z_1 = z_2$. □

Lemma 1.6.9 *In the previous framework the map $x \mapsto P_K(x)$ is a contraction.*

PROOF: We have $\langle y - P_K(y), P_K(x) - P_K(y)\rangle \leq 0$. Also $\langle x - P_K(x), P_K(y) - P_K(x)\rangle \leq 0$. Se we get $\langle y - z, P_K(x) - P_K(y)\rangle + \langle P_K(x) - P_K(y), P_K(x) - P_K(y)\rangle$ hence $\|P_K(x) - P_K(y)\|^2 \leq \langle x - y, P_K(x) - P_K(y)\rangle \leq \|x - y\| \cdot \|P_K(x) - P_K(y)\|$ so $\|P_K(x) - P_K(y)\| \leq \|x - y\|$. □

Corollary 1.6.10 (Projection onto a subspace) *Let $M \subset H$ be a closed subspace. Then $P_M : H \to M$ is linear, continuous, surjective and characterized by $P_M(x) \in M$, $x - P_M(x) \in M^{\perp}$.*

PROOF: Continuity and surjectivity are clear. By the characterization of $P_M(x)$ we have $\langle x - P_M(x), y - P_M(x)\rangle \leq 0$ for all $y \in M$. But any vector in M is of the form $y - P_M(x)$ so $\forall y_0 \in M$ we have $\langle x - P_M(x), y_0\rangle = 0$. Linearity follows from uniqueness since eg. $\langle x_1 - P_M(x_1), y\rangle = 0$ for all $y \in M$ and $\langle x_2 - P_M(x_2), y\rangle = 0$ imply $\langle x_1 + x_2 - P_M(x_1) - P_M(x_2), y\rangle = 0$ for all $y \in M$. □

PROOF: (continuation) Take $x = P_M(x) + (x - P_M(x))$. □

Definition 1.6.11 *We say that a bilinear form on H is coercive if there exists $\alpha \in \mathbb{R}_+$ such that $\forall x \in H a(x, x) \geq \alpha \|x\|^2$.*

Theorem 1.6.12 (Lax-Milgram) *Let H be a Hilbert space. Let $a : H \times H \to \mathbb{R}$ be a bilinear, continuous and coercive map. For every $T \in H^*$ there exists exactly one $xinH$ such that $a(x, y) = T(y)$ for all $y \in H$.*

Remark 1.6.13 If $a(x, y) = \langle x, y\rangle$ then the statement is equivalent to the Riesz-Fréchet theorem. If a is also symmetric then the proof is immediate since we can take $(H, \sqrt{a(\cdot, \cdot)}$ and $(H, \sqrt{\langle \cdot, \cdot\rangle})$. The Id-map between these spaces establishes an isomorphism.

PROOF: (of Lax-Milgram theorem) Consider the map $A : H \to H$ given by $x \mapsto y$ so that $a(x, \cdot) = \langle y, \cdot\rangle$. Notice that A is continuous and linear. Also notice that since $\|Ax\| \geq \langle Ax, \frac{x}{\|x\|}\rangle = \frac{1}{\|x\|} a(x, x) \geq \alpha \|x\|$ then A must be $1 - 1$ (since if $Ax = 0$ then $x = 0$ an A is linear). We want to prove that A is onto. We need to show that the range of A is the whole of H. It is again enough to show that $R(A)$ is closed and dense in H. Since $\|A \cdot\|$ is an equivalent norm we get closedness of $R(A)$. Assume that $x \in H$ such that $\forall y \in R(A)$ $\langle y, x\rangle = 0$. Then $\forall x_1 \in H$ $\langle Ax_1, x\rangle = 0$, also for $x_1 = x$ thus, $\langle Ax, x\rangle = 0$ but this is $a(x, x) = 0$ which gives $x = 0$. □

Definition 1.6.14 *Let $\{E_n\}_{n=1}^{\infty}$, be a sequence of closed subspaces of H. We say that H is a Hilbert sum of E_n and write $H = \bigoplus_{n=1}^{\infty} E_n$ if and only if*

(i) E_n's are pairwise orthogonal
(ii) $span(\bigcup_{n=1}^{\infty} E_n) = H$.

Theorem 1.6.15 *Let $H = \bigoplus E_n$. Then we have*

(i) $\forall x \in H\ x = \sum_{n=1}^{\infty} P_{E_n}(x)$
(ii) $\forall x \in H\ \|x\|^2 = \sum_{n=1}^{\infty} \|P_{E_n}(x)\|^2$ (Bessel-Parseval identity)

Moreover, the converse is true, given a sequence (x_n) in H such that $x_n \in E_n$ and $\sum_{\|} x_n\|^2 < \infty$ we have $\sum x_n$ converges and $x = \sum x_n$ has the property $x_n = P_{E_n}(x)$.

<u>Proof</u>: Define $S_n : H \to H$ $S_n(x) = \sum_{i=1}^{n} P_{E_i}(x)$. S_n are linear and continuous and $\|S_n(x)\|^2 = \sum_{i=1}^{n} \|P_{E_i}(x)\|^2 = \sum_{i=1}^{n} \langle x, P_{E_i} x\rangle$. In particular $\|S_n(x)\| \leq \|x\|$. Take $x \in H$ and fix $\epsilon > 0$. Find n such that there exists $y \in span(E_1, \ldots, E_n)$ $\|x - y\| < \epsilon$. Then $\|S_n(x) - x\| \leq \|S_n(x) - S_n(y) + y - x\| \leq 2\|y - x\| < 2\epsilon$.

If $\sum \|x_n\|^2 < \infty$ look at $S_n = \sum_{i=1}^{n} x_i$. $\|S_n - S_m\| = \|\sum_{i=n+1}^{n} x_i\| = \sqrt{\sum_{i=n+1}^{n} \|x_i\|^2} < \epsilon$ so $\{x_n\}$ is a Cauchy sequence hence it converges to some x, and as above $P_{E_n}(x) = x_n$ □

Definition 1.6.16 *Let H be a Hilbert space. A Hilbert basis of H is a sequence (e_n) such that for all m, n $\langle e_n, e_m\rangle = \delta_{mn}$ and $span(e_n)$ is dense in H.*

This essentially means that H has an orthonormal Schauder basis. We also have $H = \bigoplus span(e_n)$.

Corollary 1.6.17 *If (e_n) is a Hilbert basis of H then $x = \sum P_{span(e_n)}(x)$.*

But what is actually the projection P_{span}? $P_{span(e_n)} = \langle x, e_n\rangle e_n$. From the Bessel-Parseval equality we have that $\|x\|^2 = \sum |\langle x, e_n\rangle|^2$. Conversly, given $\alpha_n \in \ell^2$, the series $\sum \alpha_n e_n$ converges in H to some $x \in H$ and $P_{span(e_n)(x)} = \alpha_n e_n$.

Lemma 1.6.18 *Every separable Hilbert space has a Hilbert basis.*

<u>Proof</u>: Let (e_n), be done in H. Define $H_n = span(x_1, \ldots, x_n)$. Use the Gram-Schmidt procedure to find an orthonormal basis of H_1. Complement it to the basis of H_2 etc. This gives a basis since any finite subset spans an H_n and the union of H_n's is dense in H. □

Corollary 1.6.19 *Each separable Hilbert space is isometric to ℓ^2.*

1.7 The Lebesgue spaces

In the proceeding $\Omega \subset \mathbb{R}^d$ will be open, $f : \Omega \to \mathbb{R}$ will always be Lebesgue measurable and we will identify functions that differ on a set of measure zero.

Definition 1.7.1 *Let $1 \le p \le \infty$. We define the spaces $L_p(\Omega)$*
(i) for $p \in [1, \infty)$ as

$$L_p(\Omega) = \left\{ f : \Omega \to \mathbb{R}; \int_\Omega |f|^p < \infty \right\},$$

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p \right)^{1/p}.$$

(ii) for $p = \infty$

$$L_\infty(\Omega = \{ f : \Omega \to \mathbb{R}; esssup_\Omega |f| < \infty \}),$$

$$\|f\|_{L_\infty(\Omega)} = esssup_\Omega |f| = \inf\{ C > 0; |f(x)| \le C \ \forall_{a.e.} x \in \Omega \}.$$

First notice that L_p spaces are linear spaces.