Marta Lewicka, Math 8801, Fall 2007

Homework 1

1. Let $A \subset E$ be a convex, nonempty subset of a complex normed space E. Let $x_0 \notin A$. Must there always exist a nonzero $T \in E^*$ such that

$$Re\ T(x_0) \leq Re\ T(x) \qquad \forall x \in A?$$

2. Let E be a nontrivial normed space. Let $T, S : E \longrightarrow E$ be two linear operators, such that $TS - ST = Id$. Prove that they cannot be both continuous.

3. Let $\rho > 0$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ (in $\mathbf{K}$). Let E be a Banach space over $\mathbf{K}$ and let $T \in \mathcal{L}(E, E)$. Prove that if

$$\limsup_{m \to \infty} \|T^m\|^{1/m} < \rho$$

then the following operator is well defined and continuous:

$$f(T) = \sum_{n=0}^{\infty} a_n T^n \in \mathcal{L}(E, E).$$

Further, deduce that:

(i) For every $T \in \mathcal{L}(E, E)$, we have the exponential:

$$e^T = \sum_{n=0}^{\infty} (1/n!) T^n \in \mathcal{L}(E, E)$$

and $e^{T+S} = e^T e^S$ holds, provided that $TS = ST$.

(ii) For every $T \in \mathcal{L}(E, E)$, the function $A(s) = e^{sT}$ satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}s} A(s) = T A(s) = A(s) T$$

and: $A \in \mathcal{C}^\infty(\mathbf{R}, \mathcal{L}(E, E))$.

(iii) If $\|Id - T\| < 1$, then we have the logarithm:

$$\log T = -\sum_{n=0}^{\infty} (1/n)(Id - T)^n \in \mathcal{L}(E, E).$$

(iv) If $\|T\| < 1$ then the function $A(s) = \log(Id - sT)$ satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}s} A(s) = -T(Id - sT)^{-1} = -(Id - sT)^{-1} T$$

and $A \in \mathcal{C}^\infty((-1, 1), \mathcal{L}(E, E))$. Moreover: $e^{A(s)} = Id - sT$.

4. Let $\Omega_1 \subset \mathbf{R}^{n_1}$ and $\Omega_2 \subset \mathbf{R}^{n_2}$ be two open sets. Let $K : \Omega_1 \times \Omega_2 \longrightarrow \mathbf{K}$ be a Lebesgue measurable function such that, for some $1 < p, q < \infty$:

$$k := \left(\int_{\Omega_1} \left(\int_{\Omega_2} |K(x,y)|^{p'} \, \mathrm{d}y \right)^{q/p'} \mathrm{d}x \right)^{1/q} < \infty,$$

where $1/p + 1/p' = 1$. Define:

$$(Tf)(x) = \int_{\Omega_2} K(x,y) f(y) \, \mathrm{d}y.$$

Prove that $T \in \mathcal{L}(L^p(\Omega_2), L^q(\Omega_1))$ and $\|T\| \leq k$. Moreover, prove that T is a compact operator, that is: $\overline{T(B_1(0))}$ is compact in the target space $L^q(\Omega_1)$.

[T as above is called a Hilbert-Schmidt operator of integral type, with kernel K.]

5. Let $\Omega \subset \mathbf{R}^n$ be an open and bounded set. Let $K : (\overline{\Omega} \times \overline{\Omega}) \setminus \{(x,x);\ x \in \overline{\Omega}\} \longrightarrow \mathbf{K}$ be a continuous function such that, for some constants $C > 0$ and $\alpha < n$ there holds:

$$|K(x,y)| \leq \frac{C}{|x-y|^\alpha}.$$

The following is called the Schur operator with kernel K and type α:

$$(Tf)(x) = \int_{\Omega} K(x,y) f(y) \, \mathrm{d}y.$$

(i) Prove that $T \in \mathcal{L}(\mathcal{C}^0(\overline{\Omega}), \mathcal{C}^0(\overline{\Omega}))$ and that it is compact (for the definition, see the previous problem).

(ii) Prove that for every $m \in \mathbf{N}$, T^m is also a Schur operator, and find its type (as optimal, as you can).

(iii) Prove that if $1 \leq p < \infty$ and $\alpha < n/p'$, then T is a Hilbert-Schmidt operator of integral type on $L^p(\Omega)$ and it is compact from $L^p(\Omega)$ to $\mathcal{C}^0(\overline{\Omega})$.