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Homework 2

1. Let E_0 be a subspace of a Banach space, such that $\overline{E_0}$ is complemented in E. Determine if the following statement must be true. For every normed space F and every $T_0 \in \mathcal{L}(E_0, F)$, there exists $T \in \mathcal{L}(E, F)$ such that $T_{|E_0} = T$.

2. Let $T : E \longrightarrow \mathbf{K}$ be linear. Prove that:

(i) T is bounded iff its kernel is closed.

(ii) If T is unbounded then its kernel is a proper dense subspace of E.

3. Let (X, d) be metric space and define $\mathcal{A}$ to be the family of its nonempty, bounded and closed subsets. For $A_1, A_2 \in \mathcal{A}$ define the Hausdorf distance:

$$d_H(A_1, A_2) = \inf\{\epsilon > 0;\ A_1 \subset B_\epsilon(A_2) \text{ and } A_2 \subset B_\epsilon(A_1)\},$$

where we denote $B_\epsilon(A) = \{x;\ \inf_{y\in A} d(x, y) < \epsilon\}$.

(i) Prove that d_H is a metric on $\mathcal{A}$.

(ii) Let $X = \mathbf{R}^n$. Prove that the set:

$$\{A \in \mathcal{A};\ A \subset \overline{B_R(0)}\}$$

is compact in the metric space $(\mathcal{A}, d_H)$.

4. Let Ω be an open and bounded subset of $\mathbf{R}^n$. Let $0 \leq \beta < \alpha \leq 1$. Prove that the operator:

$$Id : \mathcal{C}^{0,\alpha}(\bar{\Omega}) \longrightarrow \mathcal{C}^{0,\beta}(\bar{\Omega})$$

is well defined and compact.

Can the same be said about $Id : \mathcal{C}^1(\bar{\Omega}) \longrightarrow \mathcal{C}^{0,\alpha}(\bar{\Omega})$?

5. Let Ω be an open subset of $\mathbf{R}^n$. Let $K \in L^2(\Omega\times\Omega, \mathbf{K})$ and let $T \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$ be the integral Hilbert-Schmidt operator (compare problem 4 in homework 1):

$$(Tf)(x) = \int_\Omega K(x, y)f(y)\ \mathrm{d}y.$$

Prove that $\|T\| = \|K\|_{L^2}$ if and only if:

$$K(x, y) = K_1(x) \cdot K_2(y) \qquad \forall \text{ a.e. } x, y \in \Omega,$$

for some $K_1, K_2 \in L^2(\Omega)$.