**Marta Lewicka, Math 8801, Fall 2007**

# Homework 3

**1.** Given a Banach space $E$, consider the hyperplane $H = \phi^{-1}(\{\alpha\}) \subset E^*$, for some $\phi \in E^{**} \setminus \{0\}$ and $\alpha \in \mathbf{K}$. Prove that $H$ is weak* closed if and only if $\phi = \mathrm{Ev}_T$, for some $T \in E^*$.

**2.** Let $E$ be a complex normed space and let $E^r$ be the real normed space, obtained from $E$ by restricting multiplication of vectors by scalars to $\mathbf{R} \times E$. Prove that $E$ is reflexive iff $E^r$ is reflexive.

**3.** Recall homework 1 problem 5. Using the hints from the lecture, prove that $T^m$ is Schur's oparator with type:

$$\alpha_m = \begin{cases} n - m(n-\alpha) & \text{if } m < n/(n-\alpha), \\ \epsilon & \text{for any } \epsilon > 0, \text{ if } m = n/(n-\alpha), \\ 0 & \text{if } m \geq n/(n-\alpha), \end{cases}$$

**4.** Let $X$ be a topological space. Recall that a function $\phi : X \longrightarrow [-\infty, +\infty]$ is called lowersemicontinuous (l.s.c.) iff:

$$\forall x \in X \quad \phi(x) \leq \liminf_{y \to x} \phi(y).$$

Let $E$ be a Banach space.

(i) Prove that for a convex function $\phi : E \longrightarrow (-\infty, +\infty]$, being l.s.c. with respect to strong topology on $E$ implies being l.s.c. with respect to the weak topology $\sigma(E, E^*)$. Deduce then that $\phi(x) \leq \liminf \phi(x_n)$, whenever $x_n$ converges weakly to $x$. Is the converse implication true?

(ii) Let $E$ be in addition reflexive, and let $A$ be its closed and convex subset. Let $\phi : A \longrightarrow (-\infty, +\infty]$ be convex, l.s.c., such that $\phi \not\equiv +\infty$ and satisfying:

$$\lim_{x \in A,\ \|x\| \to \infty} \phi(x) = +\infty.$$

Prove that:

$$\exists x_0 \in A \qquad \phi(x_0) = \min_{x \in A} \phi(x).$$

**5.** Let $E$ be a Banach space. Recall that a sequence $\{e_n\}_{n=1}^{\infty}$ is called a Schauder basis of $E$ iff:

$$\forall x \in E \quad \exists! \{\alpha_n\}_{n=1}^{\infty} \subset \mathbf{K} \qquad \sum_{n=1}^{\infty} \alpha_n e_n = x.$$

Given is a sequence of continuous projections $P_n : E \longrightarrow E$ with finite dimensional ranges, and satisfying:

(i) $\forall x \in E \quad \lim P_n(x) = x$,

(ii) $\forall n, m \quad P_n P_m = P_{\min\,(n,m)}$.

Define $Q_n = P_n - P_{n-1}$ (we set $P_0 = 0$).

(i) Prove that for all $n, m$ one has: $Q_n Q_m = \delta_{n,m} Q_n$. Here $\delta_{n,m}$ denotes the Kronecker delta.

(ii) Let $\{e_{n,i};\ i = 1 \dots d_n\}$ be any basis of Range $(Q_n)$. Check if the sequence:

$$e_{1,1}, e_{1,2} \dots e_{1,d_1}, e_{2,1} \dots e_{2,d_2}, e_{3,1} \dots \text{etc}$$

is a Schauder basis of $E$. If not, give a sufficient condition that it is so.