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Homework 5

1. Consider the problem on an open set $\Omega \subset \mathbb{R}^n$, with sufficiently smooth and bounded boundary:

$$(1)\qquad \begin{aligned} \frac{\partial u}{\partial t} - \Delta u = 0 &\quad \text{in } (0,\infty)\times\Omega,\\ u = 0 &\quad \text{in } (0,\infty)\times\partial\Omega,\\ u(x,0) = u_0(x) &\quad \text{in } \Omega. \end{aligned}$$

Using the maximal monotone operator theory, prove the following.

(i) Assume that $u_0 \in H_0^1(\Omega)$. Prove that the solution $u(t,x)$ to (1) satisfies $u \in \mathcal{C}([0,\infty), H_0^1(\Omega)) \cap L^2((0,\infty), H^2(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2((0,\infty), L^2(\Omega))$.

(ii) Assume that $u_0 \in H^2 \cap H_0^1(\Omega)$. Prove that the solution $u(t,x)$ to (1) satisfies $u \in \mathcal{C}([0,\infty), H^2(\Omega)) \cap L^2((0,\infty), H^3(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2((0,\infty), H^1(\Omega))$.

2. Find an explicit solution to the wave equation:

$$(2)\qquad \begin{aligned} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 &\quad \text{in } (0,\infty)\times\Omega,\\ u = 0 &\quad \text{in } (0,\infty)\times\partial\Omega,\\ u(x,0) = u_0(x) &\quad \text{in } \Omega,\\ \frac{\partial u}{\partial t}(x,0) = v_0(x) &\quad \text{in } \Omega, \end{aligned}$$

in $\Omega = \mathbb{R}$ and conclude that the related semigroup has no smoothing effect.

In all problems below assume that E is a separable Banach space.

3. Let $\{S(t)\}_{t\geq 0}$ be a $\mathcal{C}^0$-semigroup on E. Prove that:

(i) The function $[0,\infty) \ni t \mapsto \|S(t)\|$ is Borel.

(ii) The limit:

$$\omega_0 = \lim_{t\to\infty} \frac{\log\|S(t)\|}{t} \in [-\infty,\infty)$$

is well defined. It is called "the type" of the semigroup S.

(iii) $\forall \omega > \omega_0 \quad \exists M_\omega \quad \forall t \geq 0 \qquad \|S(t)\| \leq M_\omega e^{\omega t}$.

4. Let E be a Hilbert space and $\{S(t)\}_{t\geq 0}$ a $\mathcal{C}^0$-semigroup on E.

(i) Prove that for every fixed $t > 0$ and $x \in E$ the function $s \mapsto S(t-s)^* x$ is Bochner integrable on any interval $(0,\gamma)$ where $\gamma \in (0,t)$.

(ii) Prove that for a Bochner integrable function $f : (\alpha, \beta + h_0) \longrightarrow E$ there holds:

$$\lim_{h\to 0} \int_\alpha^\beta \|f(t+h) - f(t)\| \,\mathrm{d}t = 0.$$

(iii) Deduce from (i) and (ii) that for every $x \in E$ and $t > 0$ there holds:

$$\lim_{h\to 0} S(t+h)^* x = S(t)^*(x).$$

(iv) Prove that $\{S(t)^*\}_{t\geq 0}$ is a $\mathcal{C}^0$-semigroup on E^* (which we identify with E).

5. Let A be the infinitesimal generator of a $\mathcal{C}^0$-semigroup $\{S(t)\}_{t\geq 0}$ on E. Let ω_0 be the type of S (see problem 3).

(i) Prove that the resolvent $R(\lambda)$ is well defined at each $\lambda > \omega_0$ and, for a fixed $x \in E$, find the derivative of the function $\lambda \mapsto R(\lambda)(x)$.

(ii) Prove that $\frac{d^n}{d\lambda^n} R(\lambda) = (-1)^n n! R(\lambda)$.

(iii) Deduce the inverse of the Hille-Yoshida theorem, that is:

$$\forall \omega > \omega_0 \quad \exists M_\omega \quad \forall \lambda > \omega \qquad \|R(\lambda)^n\| \leq \frac{M_\omega}{(\lambda - \omega)^n} \qquad \forall n = 0, 1, 2 \dots$$

6. Prove that a necessary and sufficient condition for a closed densely defined operator A (on E) to generate a $\mathcal{C}^0$ contractive semigroup S (that is $\|S(t)\| \leq 1$ for all $t \geq 0$) is that:

$$\forall \lambda > 0 \quad \exists (\lambda \mathrm{Id} - A)^{-1} \in \mathcal{L}(E, E) \qquad \text{and} \qquad \|(\lambda \mathrm{Id} - A)^{-1}\| \leq \frac{1}{\lambda}.$$

[Hint: use the definition of the type of S from problem 3.]