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Homework 6

1. Let B be a unit ball in $\mathbb{R}^2$. For any integer n, give an example of a map $f_n : B \longrightarrow \mathbb{R}^2$, such that $\deg(f_n, B, 0) = n$.

2. Using the following outline prove the general version of Sard's theorem. Let U be an open subset of $\mathbb{R}^n$ and let $f \in \mathcal{C}^\infty(U, \mathbb{R}^m)$. Then the m-Lebesgue measure of the set of critical values: $f(C)$ is zero. Here:

$$C = \{x \in U,\ rank\ Df(x) < m\}.$$

The proof is by induction in n. For the induction step (from n to $n+1$), denote:

$$C_i = \{x \in U,\ D^k f(x) = 0 \quad \forall k \leq i\}.$$

(i) Prove that $f(C \setminus C_1)$ has measure zero. Apply change of variables to use the inductive assumption.
(ii) Prove that $f(C_i \setminus C_{i+1})$ is of measure zero.
(iii) Prove that $f(C_i)$ has measure zero, for sufficiently large i.

3. Fill in the gap of the definition of degree from class, by proving the following lemma. Let U be an open, bounded subset of $\mathbb{R}^n$ and let $f \in \mathcal{C}^2(\bar{U}, \mathbb{R}^n)$. Let $\phi \in \mathcal{C}^1_0(\mathbb{R}^n, \mathbb{R}^n)$ be such that supp $\phi \cap f(\partial U) = \emptyset$. Then the function;

$$x \mapsto (\det\, Df(x)) \cdot (\mathrm{div}\ \phi)(f(x))$$

is the divergence of some $\mathcal{C}^1_0$ vector field in U.

4. Prove the following thorem, due to Lions. Given are two Hilbert spaces V, H with their respective norms $\|\cdot\|_V$ and $\|\cdot\|_H$. Assume that V is a dense linear subspace of H and that the inclusion $i : (V, \|\cdot\|_V) \longrightarrow (H, \|\cdot\|_H)$ is continuous. Let $a(\cdot,\cdot)$ be a continuous, bilinear (or sesquilinear when the field of scalars $\mathbb{K}$ is complex) form on V, astisfying:

$$\exists \omega \in \mathbb{R} \quad \exists \alpha > 0 \quad \forall x \in V \qquad Re\ a(x,x) + \alpha \|x\|_V^2 \leq \omega \|x\|_H^2.$$

Then the operator A:

$$\mathcal{D}(A) = \{x \in V;\ a(x,\cdot) \text{ is continuous: } (V, \|\cdot\|_H) \longrightarrow \mathbb{K}\},$$

$$\forall x \in \mathcal{D}(A) \quad Ax \in H \text{ is such that } \langle Ax, y\rangle_H = a(x,y) \quad \forall y \in V$$

generates a continuous semigroup $\{S(t)\}_{t\geq 0}$ on H such that $\|S(t)\| \leq e^{\omega t}$ $\forall t \geq 0$. Use the following outline of proof.

(i) For $\lambda > \omega$ define $a_\lambda(x,y) = \lambda\langle x,y\rangle_V - a(x,y)$ for $x, y \in V$. Use Lax-Milgram theorem to show the existence of an invertible operator $\tilde{A}_\lambda \in \mathcal{L}(V,V)$ such that $a_\lambda(x,y) = \langle \tilde{A}_\lambda x, y\rangle_V$ for all $x, y \in V$.
(ii) Define A_λ for a_λ, as A was defined for a. Find the relation between A_λ and $\tilde{A}_\lambda$. That is, the domains of these operators should be equal and $A_\lambda^{-1} = \tilde{A}_\lambda^{-1} S$, where $S : H \longrightarrow V$ brings scalar product in V into scalar product in H.
(iii) Deduce that the resolvent $R(\lambda) = (\lambda \mathrm{Id} - A)^{-1} = A_\lambda^{-1}$ is well defined and $\|R(\lambda)\| \leq \frac{1}{\lambda - \omega}$. Apply the Hille-Yoshida theorem.

5. Let Ω be an open, bounded subset of $\mathbb{R}^n$, with piecewise smooth boundary. Prove that the best constant C_Ω in the following Poincare inequality:

$$\int_\Omega |f - \fint_\Omega f|^2 \leq C_\Omega \int_\Omega |\nabla f|^2 \qquad \forall f \in W^{1,2}(\Omega)$$

equals the inverse of the first (the smallest) nonzero eigenvalue of the related Neumann problem. That is, prove that $C_\Omega = 1/\alpha$, where α is the smallest nonzero solution to:

$$-\Delta u = \alpha u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{on } \partial\Omega, \qquad u \not\equiv 0. \tag{1}$$

Use the following outline of proof. Define:

$$\alpha_0 = \min \left\{ \int_\Omega |\nabla u|^2;\ u \in W^{1,2}(\Omega),\ \int u = 0,\ \int |u|^2 = 1 \right\}.$$

(i) Prove that the minimizer u_0 of the above expression exists and that if (α_0, u_0) satisfy (1) then $\alpha = \alpha_0$.
(ii) Prove that $\int (\Delta u_0 + \alpha_0 u_0)\phi = 0$, for all $\phi \in \mathcal{C}_0^1(\Omega)$ such that $\int \phi = 0$.
(iii) Prove that there must be $\Delta u_0 + \alpha_0 u_0 = const$ and that therefore $u_0 \in \mathcal{C}^\infty$.
(iv) Prove that $\int_{\partial\Omega} \frac{\partial u_0}{\partial \vec{n}} \phi = 0$ for every smooth ϕ with average 0 on Ω. Deduce that u_0 satisfies the Neumann condition.
(v) Deduce the claim $C_\Omega = 1/\alpha$.

6. Let (f, U, p) be an admissible triple (for the Brouwer degree). Let V be an open bounded set in $\mathbb{R}^n$ containing $f(\bar{U})$ and call $\{V_i\}$ the connected components of $V \setminus f(\partial U)$. Let (g, V, p) be an admissible triple. Prove the following multiplication formula:

$$\deg\ (g \circ f, U, p) = \sum_i \deg\ (f, U, p_i \in V_i) \cdot \deg\ (g, V_i, p).$$