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Homework 7

1. Derive the following properties of the Leray-Schauder degree:

(i) homotopy invariance for admissible homotopies of compact perturbations of identity,
(ii) translation invariance,
(iii) additivity property,
(iv) degree is constant on connected componets of the complement of the image of the boundary.

2. Let $f \in \mathcal{C}([0,1]^2, \mathbb{R})$ and $g \in \mathcal{C}([0,1] \times \mathbb{R}, \mathbb{R})$. Assume that g is bounded. Prove that there exists a continuous function $u \in \mathcal{C}([0,1], \mathbb{R})$ such that;

$$u(s) = \int_0^1 f(s,t) g(t, u(t)) \, \mathrm{d}t \quad \forall s \in [0,1].$$

[Hint. Use compactness.]

3. Let $T \in \mathcal{L}(E, E)$ be a operator on a Banach space, with a compact power that is: $T^n \in \mathcal{K}(E, E)$, for some natural number n. Prove that:

(i) the kernel of $\mathrm{Id} - T$ has finite dimension,
(ii) $\exists k \quad \forall i > k \quad \ker\, (\mathrm{Id} - T)^i = \ker\, (\mathrm{Id} - T)^k$.
(iii) the range of $\mathrm{Id} - T$ is closed.

[Hint: Write $\mathrm{Id} - T^n = (\mathrm{Id} - T)(\mathrm{Id} + T + T^2 + \dots T^{n-1})$.]

4. Prove the following Lomonosov's theorem. Let $T, S \in \mathcal{L}(E, E)$ be two linear continuous operators on a Banach space E. Assume that $TS = ST$ and that T is nonzero and compact. Then S has a nontrivial invariant subspace, that is $F \subset E$ such that $S(F) \subset F$ and $F \neq E, \{0\}$.

5. Let $f = (f_1, f_2) \in \mathcal{C}^2(\Omega, \mathbb{R}^2)$ be a continuous up to the boundary vector field on an open, bounded set $\Omega \subset \mathbb{R}^2$. Assume that $\det Df$ does not change sign in Ω and let $x_0 \in \Omega$ be such that $\det Df(x_0) \neq 0$. Prove that there exists $x_1 \in \partial\Omega$ such that:

$$f_1(x_0) = f_1(x_1).$$

[Hint. Argueing by contradiction, consider an open set $\Omega_1 = f_1^{-1}(\alpha, \beta)$ containing x_0 and compactly contained in Ω. Use Green's theorem to prove that $\int_{\Omega_1} \det Df = 0$. For noting the regularity of $\partial\Omega_1$, use general version of Sard's lemma.