MATH 8302: Manifolds & Topology
Homework 4

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March 25, 2019

The book referenced throughout is [1].

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1 5.12.

Künneth formula for compact cohomology. The Künneth formula for compact cohomology states that for any manifolds M and N having a finite good cover,

$$H_c^*(M \times N) = H_c^*(M) \otimes H_c^*(N).$$

(a) In case M and N are orientable, show that this is a consequence of Poincaré duality and the Künneth formula for de Rham cohomology.

(b) Using the Mayer-Vietoris argument, prove the Künneth formula for compact cohomology for any M and N having a finite good cover.

Solution. (a) Since M, N have finite good covers, it follows that their cohomologies and compact cohomologies are finite-dimensional, whence Poincaré duality does tell us that

$$H_c^q(X) \simeq H^{n-q}(X), \qquad \text{for each } q \in \mathbb{N}_0,$$

where $X = M, N$. Let m be the dimension of M and n the dimension of N. Then, for each integer $k = 0, \ldots, m+n$, we have that

$$H_c^k(M \times N) = \Big(H^{m+n-k}(M \times N)\Big)^* = \Big(\bigoplus_{p+q=m+n-k} H^p(M) \otimes H^q(N)\Big)^*$$

$$= \bigoplus_{p+q=m+n-k} (H^p(M))^* \otimes (H^q(N))^* = \bigoplus_{p+q=m+n-k} H_c^{m-p}(M) \otimes H_c^{n-q}(N)$$

$$= \bigoplus_{s+t=k} H_c^s(M) \otimes H_c^t(N),$$

where we used Poincaré duality, then the Kunneth formula for De Rham cohomology, then the commutativity of the dual operator $(\cdot)^*$ with direct sum and tensor product, then Poincaré duality, and finally a change of variables $s = m - p, t = n - q$.

(b) We follow the Mayer-Vietoris argument. The natural projections $\pi : M \times N \to M$ and $\rho : M \times N \to N$ give rise to a map on forms with compact support

$$\omega \otimes \phi \mapsto \pi^*\omega \wedge \rho^*\phi.$$

We have that $\pi^*\omega \wedge \rho^*\phi$ has compact support in $M \times N$. Hence we have the pushforward map in compact cohomology

$$\psi : H_c^*(M) \otimes H_c^*(N) \to H_c^*(M \times N).$$

We are done as soon as we show that ψ is an isomorphism, which we now intend to prove. Let U and V be open sets in M and let n be a fixed integer. From the Mayer-Vietoris sequence

$$\cdots \to H_c^p(U \cap V) \to H_c^p(U) \oplus H_c^p(V) \to H_c^p(U \cup V) \to \cdots$$

we get an exact sequence by tensoring with H_c^{n-p},

$$\cdots \to H_c^p(U \cap V) \otimes H_c^{n-p}(N) \to (H_c^p(U) \oplus H_c^p(V)) \otimes H_c^{n-p}(N)$$
$$\to H^p(U \cup V) \otimes H_c^{n-p}(N) \to \cdots,$$

since tensoring with a vector space preserves exactness. Summing over all integers p

yields the exact sequence

$$\cdots \to \bigoplus_{p=0}^{n} H_c^p(U \cap V) \otimes H_c^{n-p}(N)$$
$$\to \bigoplus_{p=0}^{n} \Big((H_c^p(U) \otimes H_c^{n-p}(N)) \oplus (H_c^p(V) \otimes H_c^{n-p}(N)) \Big)$$
$$\to \bigoplus_{p=0}^{n} H_c^p(U \cup V) \otimes H_c^{n-p}(N) \to \cdots .$$

The following diagram is commutative

$$\begin{array}{ccccc}
\bigoplus_{p=0}^{n} H_c^p(U \cap V) \otimes H_c^{n-p}(N) & \longrightarrow & \bigoplus_{p=0}^{n} (H_c^p(U) \otimes H_c^{n-p}(N)) \oplus (H_c^p(V) \otimes H_c^{n-p}(N)) & \longrightarrow & \bigoplus_{p=0}^{n} H^p(U \cup V) \otimes H_c^{n-p}(N) \\
\downarrow \psi & & \downarrow \psi & & \downarrow \psi \\
H_c^n((U \cap V) \times N) & \longrightarrow & H_c^n(U \times N) \oplus H_c^n(V \times N) & \longrightarrow & H_c^n((U \cup V) \times N).
\end{array}$$

Since M is an $m-$manifold with finite good cover, each of $U, V, U \cap V$ is diffeomorphic to $\mathbb{R}^m$. Note that $H_c^k(\mathbb{R}^m) \simeq 0$ for all $k \neq m$, and $H_c^m(\mathbb{R}^m) \simeq \mathbb{R}$ (see p.46). Hence, if $n \geq m$, then

$$\bigoplus_{p=0}^{n} H_c^p(\mathbb{R}^m) \otimes H_c^{n-p}(N) \cong \mathbb{R} \otimes H_c^{n-m}(N) \cong H_c^{n-m}(N) \cong H_c^n(\mathbb{R}^m \times N),$$

where we used Proposition 4.7 in the last step. Hence the Kunneth formula is verified for U, V, and $U \cap V$. By the Five lemma, then the Kunneth formula is also true for $U \cup V$. enough to show that ψ is an isomorphism on $U, V, U \cap V$. The Kunneth formula now follows by induction on the cardinality of a good cover, as in the proof of Poincaré duality. □

2 5.16.

The ray and the circle in $\mathbb{R}^2 \backslash \{0\}$. Let x, y be the standard coordinates and r, θ the polar coordinates on $\mathbb{R}^2 \backslash \{0\}$.

(a) Show that the Poincaré dual of the ray $\{(x, 0) : x > 0\}$ in $\mathbb{R}^2 \backslash \{0\}$ is $d\theta / 2\pi$ in $H^1(\mathbb{R}^2 \backslash \{0\})$.

(b) Show that the closed Poincaré dual of the unit circle in $H^1(\mathbb{R}^2 \backslash \{0\})$ is 0, but the compact Poincaré dual is the nontrivial generator $\rho(r) dr$ in $H_c^1(\mathbb{R}^2 \backslash \{0\})$ where $\rho(r)$ is a bump function with total integral 1.

Solution. (a). Let $M = \mathbb{R}^2\backslash\{0\}$ and $S = \{(x,0) : x > 0\}$, which is a closed oriented submanifold of dimension 1. Let $i : S \to M$ be the inclusion map. We need to show that for any $\omega \in H^1_c(M)$, we have that

$$\int_S i^*\omega = \int_M \omega \wedge \frac{d\theta}{2\pi}.$$

So let $\omega \in H^1_c(M)$, so that there exist $f, g \in C^\infty_c(M)$ such that $\omega = f(r,\theta)dr + g(r,\theta)d\theta$. Now, $d\omega = 0$ because ω must be closed, and hence it follows that $\frac{\partial f}{\partial \theta} = \frac{\partial g}{\partial r}$. Integrating this identity over r from 0 to ∞ yields that

$$\frac{\partial}{\partial\theta}\Big(\int_0^\infty f(r,\theta)\,dr\Big) = \int_0^\infty \frac{\partial g(r,\theta)}{\partial r}\,dr = 0,$$

where in the last equality we used the Fundamental Theorem of Calculus and the fact that g is compactly supported in M. Thus the quantity $\int_0^\infty f(r,\theta)\,dr$ is a constant in θ. Hence,

$$\begin{aligned}\int_M \omega \wedge \frac{d\theta}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^\infty f(r,\theta)\,dr\Big)\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^\infty f(r,0)\,dr\Big)\,d\theta \\ &= \int_0^\infty f(r,0)\,dr = \int_S f|_S\,dx = \int_S i^*\omega,\end{aligned}$$

as desired.

(b) Let $M = \mathbb{R}^2\backslash\{0\}$ and S is the unit circle, which is a closed oriented submanifold of dimension 1. Let $i : S \to M$ be the inclusion map. To show that 0 is the closed Poincaré dual of S, we have to prove that for any $\omega \in H^1_c(M)$, we have that

$$\int_S i^*\omega = 0.$$

So let $\omega \in H^1_c(M)$, so that there exist $f, g \in C^\infty_c(M)$ such that $\omega = f(r,\theta)dr + g(r,\theta)d\theta$. Now, $d\omega = 0$ because ω must be closed, and hence it follows that $\frac{\partial f}{\partial \theta} = \frac{\partial g}{\partial r}$. Integrating this identity over θ from 0 to 2π yields easily that the quantity $\int_0^{2\pi} g(r,\theta)\,d\theta$ is a constant in r. Since for all r and all θ large enough, $g(r,\theta \equiv 0$ since g is compactly supported, we conclude that $\int_0^{2\pi} g(r,\theta)\,d\theta = 0$, for some (and hence, for every) $r > 0$. Consequently,

$$\int_S i^*\omega = \int_0^{2\pi} g(1,\theta)\,d\theta = 0,$$

as claimed.

We now purport to show that $\rho(r)\,dr$ is the compact Poincaré dual of S, where $\rho(r)$ is a bump function such that $\int_0^\infty \rho(r)\,dr = 1$. To do so, we have to prove that for any $\omega \in H^1(M)$, we have that

$$\int_S i^*\omega = \int_M \omega \wedge (\rho(r)\,dr).$$

So let $\omega \in H^1(M)$, so that there exist $f, g \in C^\infty(M)$ such that $\omega = f(r,\theta)dr + g(r,\theta)d\theta$. Now, $d\omega = 0$ because ω must be closed, and hence it follows that $\frac{\partial f}{\partial \theta} = \frac{\partial g}{\partial r}$. Integrating this identity over θ from 0 to 2π yields easily that the quantity $\int_0^{2\pi} g(r,\theta)\, d\theta$ is a constant in r. Thus, we observe that

$$\begin{aligned}\int_S i^*\omega &= \int_0^{2\pi} g(1,\theta)\, d\theta = \Big(\int_0^{2\pi} g(1,\theta)\, d\theta\Big)\Big(\int_0^\infty \rho(r)\, dr\Big) \\ &= \int_0^\infty \rho(r)\Big(\int_0^{2\pi} g(1,\theta)\, d\theta\Big)\, dr = \int_0^\infty \rho(r)\Big(\int_0^{2\pi} g(r,\theta)\, d\theta\Big)\, dr \\ &= \int_M \Big[f(r,\theta)dr + g(r,\theta)d\theta\Big] \wedge (\rho(r)\, dr) = \int_M \omega \wedge (\rho(r)\, dr),\end{aligned}$$

where in the second equality we used the fact that the integral of ρ is 1, in the fifth equality we used that $drdr = 0$ and Fubini's Theorem which is applicable since ρ is non-negative, smooth, and has bounded support in M (since it is a bump function), and $|g|$ is bounded in the support of ρ. The claim follows. □

3 6.2.

Show that two vector bundles on M are isomorphic if and only if their cocycles relative to some open cover are equivalent.

Solution. (Only if). Let $(E,\pi), (E',\pi')$ be two vector bundles over M which are isomorphic, so that there is a vector bundle isomorphism $f : E \to E'$. Let (U_α, ϕ_α) be the open cover of M with the corresponding trivializations for E, afforded by its definition. Then $(U_\alpha, \phi_\alpha \circ f^{-1})$ is an open cover of M together with trivializations $\phi'_\alpha := \phi_\alpha \circ f^{-1} : E'|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ for some n. Fix α, β and $x \in U_\alpha \cap U_\beta$. Note that, in this case via our construction,

$$g'_{\alpha\beta}(x) = \phi'_\alpha \phi'^{-1}_\beta(x) = \phi_\alpha f^{-1} f \phi_\beta^{-1}(x) = \phi_\alpha \phi_\beta^{-1}(x) = g_{\alpha\beta}(x),$$

so that $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are equivalent, but we are not technically done yet because (E',π') may be a priori endowed with different trivializations than ϕ'_α. So let $\{\phi''_\alpha\}$ be any collection of trivializations with which E' is endowed over the open cover U_α. Then we may use Lemma 6.1 to see that $g''_{\alpha\beta}$ is equivalent with $g'_{\alpha\beta}$. Since equivalence is transitive, we thus have that $g_{\alpha\beta}$ is equivalent with $g''_{\alpha\beta}$, as desired.

(If). Now fix an open cover $\{U_\alpha\}$ of M. Let $(E,\pi), (E',\pi')$ be two vector bundles over M, let $\phi_\alpha, \phi_{\alpha'}$ be the respective trivializations over $\{U_\alpha\}$, and let $g_{\alpha\beta}, g'_{\alpha\beta}$ be the respective cocycles. By hypothesis, there exist invertible maps $\lambda_\alpha : U_\alpha \to GL(n,\mathbb{R})$ such that

$$g_{\alpha\beta} = \lambda_\alpha g'_{\alpha\beta} \lambda_\beta^{-1}, \qquad \text{on } U_\alpha \cap U_\beta,$$

(here, λ_β^{-1} is the inverse matrix to λ_β, not the inverse map of λ_β). For each U_α, let $f_\alpha : E|_{U_\alpha} \to E'|_{U_\alpha}$ be the map given by

$$f_\alpha := \phi_\alpha'^{-1} \circ (\lambda_\alpha^{-1} \cdot \phi_\alpha).$$

It is instructive to chase the map of f_α. Let $x \in U_\alpha$ and $\zeta \in \pi^{-1}(x)$. We use the notation $\vec{\phi}$ for the second component of the map ϕ (the one that maps into $\mathbb{R}^n$). Then

$$\zeta \overset{\phi_\alpha}{\longmapsto} (x, \vec{\phi}_\alpha \zeta) \overset{\lambda_\alpha^{-1}}{\longmapsto} (x, \lambda_\alpha^{-1}(x)\vec{\phi}_\alpha \zeta) \overset{\phi_\alpha'^{-1}}{\longmapsto} \pi'^{-1}\Big(x, \vec{\phi}_\alpha'^{-1}\lambda_\alpha^{-1}(x)\vec{\phi}_\alpha\zeta\Big).$$

Thus we see that f_α is a fiber-preserving smooth map that is linear on corresponding fibers. Now, if $x \in U_\alpha \cap U_\beta$ and $\zeta \in \pi^{-1}(x)$, then $f_\alpha(\zeta)$ is given above, while similarly we have that

$$f_\beta(\zeta) = \pi'^{-1}\Big(x, \vec{\phi}_\beta'^{-1}\lambda_\beta^{-1}(x)\vec{\phi}_\beta\zeta\Big).$$

Hence we see that $f_\alpha(x) = f_\beta(x)$ if and only if

$$\vec{\phi}_\alpha'^{-1}\lambda_\alpha^{-1}\vec{\phi}_\alpha = \vec{\phi}_\beta'^{-1}\lambda_\beta^{-1}\vec{\phi}_\beta,$$

which in turn occurs if and only if

$$\vec{\phi}_\alpha\vec{\phi}_\beta^{-1} = \lambda_\alpha\vec{\phi}_\alpha'\vec{\phi}_\beta'^{-1}\lambda_\beta^{-1},$$

which is guaranteed by the hypothesis. It follows that the map

$$f := f|_{U_\alpha}, \qquad \text{on each } U_\alpha$$

is a well-defined vector bundle isomorphism. □

4 6.10.

Compute $\mathrm{Vect}_k(S^1)$.

Solution. Recall that $\mathrm{Vect}_k(S^1)$ is the isomorphism classes of rank k real vector bundles over S^1. Let (E, π) be a vector bundle over S^1, and let $f : [0,1] \to S^1$ be given by $t \mapsto e^{2\pi i t}$. Then $f^{-1}E$ is a vector bundle over $[0,1]$. Since $[0,1]$ is contractible, by Corollary 6.9 we have that $f^{-1}E$ is the trivial bundle $[0,1] \times \mathbb{R}^k$. Now consider all smooth maps $[0,1] \to S^1$. There are exactly two homotopy classes of such maps, corresponding to $[f]$ and $[-f] = [e^{-2\pi i t}]$, whence by Theorem 6.8 we conclude that for each $k \in \mathbb{N}$, there are two isomorphism classes in $\mathrm{Vect}_k(S^1)$, corresponding to $[f^{-1}E]$ and $[(-f)^{-1}E]$. □

5 6.14.

Show that if E is an oriented vector bundle, then $\pi_*\omega_\alpha = \pi_*\omega_\beta$ on $U_\alpha \cap U_\beta$. Hence $\{\pi_*\omega_\alpha\}_{\alpha\in I}$ piece together to give a global form $\pi_*\omega$ on M. Furthermore, this definition is independent of the choice of the oriented trivialization for E.

Solution. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ be the coordinate functions on U_α and U_β, and $t = (t_1, \ldots, t_n), u = (u_1, \ldots, u_n)$ the fiber coordinates on $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\beta)$ respectively. Fix $\omega \in \Omega^*_{cv}(E)$ and recall that $\omega_\alpha := \omega|_{\pi^{-1}(U_\alpha)}$. By chasing the inclusion maps $j_\alpha : U_\alpha \cap U_\beta \hookrightarrow U_\alpha, j_\beta : U_\alpha \cap U_\beta \hookrightarrow U_\beta, i_\beta : U_\beta \hookrightarrow M, i_\alpha : U_\alpha \hookrightarrow M$ and observing that $i_\alpha j_\alpha = i_\beta j_\beta$ is the same inclusion map, we deduce that

$$\omega_\alpha|_{\pi^{-1}(U_\alpha\cap U_\beta)} = \omega_\beta|_{\pi^{-1}(U_\alpha\cap U_\beta)}. \tag{5.1}$$

A form $\omega \in \Omega^*_{cv}(E)$ is locally of type (I) or (II). If ω_α is of type (I), then $\pi_*\omega_\alpha$ is the zero form, and in particular, it is identically 0 on $U_\alpha \cap U_\beta$, whence by (5.1), we have that $\pi_*\omega_\beta = \pi_*\omega_\alpha = 0$ on $U_\alpha \cap U_\beta$.

Hence, we may now assume that both $\omega_\alpha, \omega_\beta$ are of type (II). Then there exist (see p.61) forms ψ and τ on M, and f, g compactly supported functions for each fixed $\zeta \in M$, such that

$$\omega_\alpha = (\pi^*\psi) f(x,t)\, dt, \qquad \omega_\beta = (\pi^*\tau) g(y,u)\, du.$$

Owing to (5.1), it follows that

$$(\pi^*\psi|_{U_\alpha\cap U_\beta}) f(x(\zeta),t)\, dt = (\pi^*\tau|_{U_\alpha\cap U_\beta}) g(y(\zeta),u)\, du, \qquad \text{for each } \zeta \in U_\alpha \cap U_\beta.$$

Observe the calculation

$$\begin{aligned}\pi_*\omega_\alpha|_{U_\alpha\cap U_\beta} = \psi|_{U_\alpha\cap U_\beta} \int_{\mathbb{R}^n} f(x,t)\, dt &= \int_{\mathbb{R}^n} (\pi^*\psi|_{U_\alpha\cap U_\beta}) f(x,t)\, dt \\ &= \int_{\mathbb{R}^n} (\pi^*\tau|_{U_\alpha\cap U_\beta}) g(y,u)\, du = \tau|_{U_\alpha\cap U_\beta} \int_{\mathbb{R}^n} g(y,u)\, du \\ &= \pi_*\omega_\beta|_{U_\alpha\cap U_\beta},\end{aligned}$$

as desired. It is clear then that it does not matter which oriented trivialization we choose for E. □

6 6.20.

Using a Mayer-Vietoris argument as in the proof of the Thom isomorphism (Theorem 6.17), show that if $\pi : E \to M$ is an orientable rank n bundle over a manifold M of finite type, then

$$H^*_c(E) \simeq H^{*-n}_c(M).$$

Solution. Our program is to show that $\pi_* : H_c^*(E) \to H_c^{*-n}(M)$ is an isomorphism. We adapt the proof of Theorem 6.7. Let U and V be open subsets of M. Using a partition of unity from the base M we see that

$$0 \to \Omega_c^*(E|_{U\cap V}) \to \Omega_c^*(E|_U) \oplus \Omega_c^*(E|_V) \to \Omega_c^*(E|_{U\cup V}) \to 0$$

is exact, as in Proposition 2.7. So we have the diagram of Mayer-Vietoris sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_c^*(E|_{U\cap V}) & \longrightarrow & H_c^*(E|_U) \oplus H^*(E|_V) & \longrightarrow & H_c^*(E|_{U\cup V}) & \xrightarrow{d^*} & H_c^{*+1}(E|_{U\cap V}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & \\
\cdots \longrightarrow & H_c^{*-n}(U\cap V) & \longrightarrow & H_c^{*-n}(U) \oplus H_c^{*-n}(V) & \longrightarrow & H_c^{*-n}(U\cup V) & \xrightarrow{d^*} & H^{*+1-n}(U\cap V) & \longrightarrow \cdots
\end{array}.$$

The above diagram is clearly commutative. By Corollary 6.9, if U is diffeomorphic to $\mathbb{R}^n$, then $E|_U$ is the trivial bundle, so that by the Poincaré lemma for compact support we have that $\pi_* : H_c^*(E|_U) \to H_c^{*-n}(U)$ is an isomorphism. By the Five Lemma, since the desired conclusion holds for U, V, and $U \cap V$, then it holds for $U \cup V$. The proof now proceeds by induction on the cardinality of a good cover for the base, as in the proof of Poincaré duality. □

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer.