MFM 5022:
HOMEWORK #9

Question 0.1:

Question 0.2:

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a)

$$\int_0^t X_s^n \, dW_s = \sum_{k=0}^{n-1} \int_{\frac{k}{n}t}^{\frac{k+1}{n}t} X_s^n \, dW_s = \sum_{k=0}^{n-1} \int_{\frac{k}{n}t}^{\frac{k+1}{n}t} e^{\frac{\alpha k t}{n}} \, dW_s$$

$$= \sum_{k=0}^{n-1} e^{\frac{\alpha k t}{n}} \int_{\frac{k}{n}t}^{\frac{k+1}{n}t} 1 \, dW_s = \sum_{k=0}^{n-1} e^{\frac{\alpha k t}{n}} (W_{\frac{k+1}{n}t} - W_{\frac{k}{n}t}).$$

b) X_s^n **is deterministic, so we have**

$$E(\int_0^t (x_s^n)^2 ds) = \int_0^t (x_s^n)^2 ds = \sum_{k=0}^{n-1} \int_{\frac{k}{n}t}^{\frac{k+1}{n}t} (X_s^n)^2 \, ds = \sum_{k=0}^{n-1} \int_{\frac{k}{n}t}^{\frac{k+1}{n}t} e^{2\alpha k t/n} \, ds = \sum_{k=0}^{n-1} e^{2\alpha k t/n} \frac{t}{n}.$$

$$= \frac{t}{n} \sum_{k=0}^{n-1} \left(e^{2\alpha t/n}\right)^k = \frac{t}{n} \frac{e^{2\alpha t} - 1}{e^{2\alpha t/n} - 1}$$

c) Yes.

$$\int_0^t (e^{\alpha s} - x_s^n)^2 ds = \sum_{k=0}^{n-1} \int_{\frac{k}{n}t}^{\frac{k+1}{n}t} (e^{\alpha s} - e^{\alpha k t/n})^2 \, ds \le \sum_{k=0}^{n-1} \int_{\frac{k}{n}t}^{\frac{k+1}{n}t} (e^{\alpha(k+1)t/n} - e^{\alpha k t/n})^2 \, ds,$$

the inequality holds because $(e^{\alpha s} - e^{\alpha k t/n})^2 \le (e^{\alpha(k+1)t/n} - e^{\alpha k t/n})^2$ **for any** $s \in (kt/n, (k+1)t/n)$**. Next, since** $|e^{\alpha(k+1)t/n} - e^{\alpha k t/n}| \le |\alpha| e^{\alpha \xi}((k+1)t/n - kt/n) \le Nt/n$ **where** $N = |\alpha|(e^{\alpha t} + 1)$ **is a constant, for some** $\xi \in (kt/n, (k+1)t/n)$**, it follows**

$$\sum_{k=0}^{n-1} \int_{\frac{k}{n}t}^{\frac{k+1}{n}t} (e^{\alpha(k+1)t/n} - e^{\alpha k t/n})^2 \, ds \le \sum_{k=0}^{n-1} \int_{\frac{k}{n}t}^{\frac{k+1}{n}t} (Nt/n)^2 \, ds = N^2 t^3/n^2 \to 0,$$

as $n \to \infty$**. (What we proved is the uniform convergence.)**

A different approach: It is easy to check that X_s^n converges to $e^{\alpha s}$ pointwise which in turn implies $|e^{\alpha s} - x_s^n|^2 \leq e^{2\alpha s}$ converges to 0 pointwise, and $|e^{\alpha s} - x_s^n|^2 \leq e^{2\alpha s}$, So we can apply Dominated Convergence Theorem.

d) First, we have

$$E(\int_0^t e^{\alpha s}\, dW_s) = 0,$$

the variance of $\int_0^t e^{\alpha s}\, dW_s$ is therefore

$$E((\int_0^t e^{\alpha s}\, dW_s)^2)$$

It is worth noting that if a sequence of Gaussian variables Y_n converges to Y, then Y is also Gaussian, moreover, the variance of Y is the limit of the variance of Y_n. Let $Y_n = \int_0^t X_s^n\, dW_s$, by Definition, a) , Fubini's theorem, independence of increment of Brownian Motion,

$$\int_0^t e^{\alpha s}\, dW_s = \lim_{n\to\infty} Y_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} e^{\frac{\alpha k t}{n}} (W_{\frac{k+1}{n}t} - W_{\frac{k}{n}t})$$

and

$$\begin{aligned}
E((\int_0^t e^{\alpha s}\, dW_s)^2) &= \lim_{n\to\infty} E(Y_n^2) = \lim_{n\to\infty} E\big[\sum_{k=0}^{n-1} e^{\frac{\alpha k t}{n}} (W_{\frac{k+1}{n}t} - W_{\frac{k}{n}t})\big]^2 \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} E\big[e^{\frac{\alpha k t}{n}} (W_{\frac{k+1}{n}t} - W_{\frac{k}{n}t}) e^{\frac{\alpha l t}{n}} (W_{\frac{l+1}{n}t} - W_{\frac{l}{n}t})\big] \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} e^{\frac{\alpha(k+l)t}{n}} E[(W_{\frac{k+1}{n}t} - W_{\frac{k}{n}t})(W_{\frac{l+1}{n}t} - W_{\frac{l}{n}t})] \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} e^{\frac{\alpha(k+l)t}{n}} \delta_{k,l} E[(W_{\frac{k+1}{n}t} - W_{\frac{k}{n}t})]^2 \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1} e^{\frac{2\alpha k t}{n}} (\frac{k+1}{n}t - \frac{k}{n}t) = \lim_{n\to\infty} \sum_{k=0}^{n-1} e^{\frac{2\alpha k t}{n}} \frac{t}{n} \\
&= \lim_{n\to\infty} E(\int_0^t (x_s^n)^2 ds) = E(\int_0^t e^{2\alpha s} ds) = \frac{e^{2\alpha t} - 1}{2\alpha}.
\end{aligned}$$
