BRIEF ARTICLE

1. SOLUTION TO HOMEWORK 1

use equations 15.11 and 15.12 from the textbook. for the investor in Japan, the value of the call is

$$c = S_0 e^{-r_D T} N(d_1) - K e^{-r_J T} N(d_2),$$

where

$$d_1 = \frac{\ln(S_0/K) + (r_J - r_D + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\ln(S_0/K) + (r_J - r_D - \sigma^2/2)T}{\sigma\sqrt{T}},$$

r_D denotes the risk-free interest rate in US currency, r_J denotes the risk-free interest rate in Japanese currency, S_0 is the value of one dollar at time $t = 0$. In our case $S_0 = 90$ Yen, and K is the strike price, which equals S_0 for an ATM call. Hence (in Yen)

$$c = 90 e^{-r_D T} N(d_1) - 90 e^{-r_J T} N(d_2),$$

where

$$d_1 = \frac{(r_J - r_D + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_1 = \frac{(r_J - r_D - \sigma^2/2)T}{\sigma\sqrt{T}}$$

similarly for US investor, (in USD)

$$p = e^{-r_D T} N(-\tilde{d}_2) - e^{-r_J T} N(-\tilde{d}_1)$$

where

$$\tilde{d}_2 = -d_1, \quad , -\tilde{d}_1 = d2.$$

by comparing the equations we see that the value of an ATM call on one dollar is the same as the value of an ATM put.

2. SOLUTION TO HOMEWORK 2

Suppse that the returns of two stocks (A and B) have the same standard derviation (30%) and correlation matrix:

$$M = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}.$$

a) Computer the eigenvalues and eigenvectors of the matrix.

Solving $\det(M - \lambda I) = 0$, we get $\lambda_1 = 0.3$ and $\lambda_2 = 1.7$. Substituting the eigenvector in $(M - \lambda I)\vec{X} = 0$. we get the the eigenvector corresponding to 0.3 is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ and the eigenvector corresponding to 0.7 is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

b) Describe how to set up a Monte Carlo simulation for stocks A and B that takes into account their volatilities and correlation.

We assume that stocks A and B follow the following processes

$$dA = rA\,dt + 0.3A\,dX_1, \tag{1}$$

$$dB = rB\,dt + 0.3B\,dX_2, \tag{2}$$

with correlation matrix $\begin{pmatrix}1 & 0.7\\0.7 & 1\end{pmatrix}$.

is a symmetric matrix whose eigenvalue decomposition is

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1.7 & 0\\0 & 0.3\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}.$$

We can generate Brownian motions $dX_1\,dX_2$ with correlation matrix M by letting

$$\begin{pmatrix}dX_1\\dX_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}\sqrt{1.7} & 0\\0 & \sqrt{0.3}\end{pmatrix}\begin{pmatrix}dW_1\\dW_2\end{pmatrix},$$

where W_1 and W_2 are two standard normal variables. Multiplying out, we get

$$dX_1 = \sqrt{0.85}\,dW_1 + \sqrt{0.15}\,dW_2 \tag{3}$$

$$dX_2 = \sqrt{0.85}\,dW_1 - \sqrt{0.15}\,dW_2 \tag{4}$$

By solving equations (1), (2), we can set up a Monte Carlo Simulation for stocks A and B. Define A(t) as the stock price of A at time t, B(t) as the stock stock price of B at time t. W_1, W_2 are two independent normal variables. Then

$$A(t) = A(t - \Delta t)exp((r - \frac{.3^2}{2})\Delta t + 0.3\sqrt{\Delta t}X_1),$$

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3. Solution to homework 3

3.1. Explan why the implicit method is called "Implicit".

3.1. Explan why the implicit method is called "Implicit". Explicit methods calculate the state of a system at a later time from the state of a system at the current time. Mathematically, if $Y(t)$ is the state of that system, and $Y(t+\Delta t)$ is the state of that system at a later time then, for an explicit method,

$$Y(t+\Delta t) = F(Y(t)), \tag{5}$$

where F is a function of the state Y. In other words, we can solve for $Y(t+\Delta t)$ explicitly. when we don't have equation of the form (5) but we can still find a method where both $Y(t)$ and $Y(t+\Delta t)$ are related via an equation of the form

$$G(Y(t), Y(t+\Delta t)) = 0 \tag{6}$$

The method is called "implicit" because we can no long solve for $Y(t+\Delta t)$ explicitly.

3.2. Suppose that you have a random number generator for a uniform distribution: a) How do you use it to generate a normal variable ?

We can use Box-Muller method. Let x_1 and x_2 be random variables uniformly distributed on the interval $(0,1)$. Then

$$X_1 = \sqrt{-2\ln x_1}\cos(2\pi x_2), \quad X_2 = \sqrt{-2\ln x_1}\cos(2\pi x_2)$$

are two independent standard normal random variables.

b) How do you simulate a pair of normal variables X and Y with correlation ρ?

First simulate too independent standard normal variables X_1 and X_2 as in a). We can prove that variables $X = X_1, Y = \rho X_1 + \sqrt{1-\rho^2}X_2$ are standard normal with correlation ρ. First, Y is normal since sum of two normal variables is a normal variable. Next, $EY = \rho EX_1 + \sqrt{1-\rho^2}EX_2 = 0$ and $var(Y) = \rho^2 EX_1^2 + (1-\rho^2)EX_2^2$ since X_1 and X_2 are independent. It follows,

$$Corr(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \rho.$$

4. Solution to homework 4

1) A random variable X follows an Exponential distribution with parameter λ if its density is given by

$$f(x) = \lambda e^{-\lambda x} 1_{x>0}.$$

Suppose that you obtain a sample $x_1, \cdots, x_n$ from the random variable X. How would you estimate λ using maximum likelihood?

The likelihood

$$\mathcal{L}(\lambda|x_1,\cdots,x_n) = f(x_1,\cdots,x_n|\lambda) = \prod_i f(x_i|\lambda) = \lambda^n e^{-\lambda\sum_i x_i} 1_{\{x_1>0\cdots,x_n>0\}}$$
$$= e^{n\ln\lambda-\lambda\sum_i x_i} 1_{\{x_1>0\cdots,x_n>0\}}$$

the average log-likelihood is

$$\hat{l} = \frac{\ln\mathcal{L}(\lambda|x_1,\cdots,x_n)}{n} = \ln\lambda - \lambda\frac{\sum_i x_i}{n}$$

$$\frac{d\hat{l}}{d\lambda} = \frac{1}{\lambda} - \frac{\sum_i x_i}{n}$$

As a function of λ, $\hat{l}$ increases for $\lambda < \frac{\sum_i x_i}{n}$ and decreases when $\lambda > \frac{\sum_i x_i}{n}$. hence it take maximum at $\lambda = \frac{\sum_i x_i}{n}$. The maximum likelihood estimator of λ is $\frac{\sum_i x_i}{n}$.