A STUDY OF THE KIRCHHOFF THEORY OF ELASTIC PRE-STRAINED SHELLS

HUI LI

ABSTRACT. Applying methods of Γ-convergence, we introduce and justify a variant of Kirchhoff theory for thin 3d shells, valid in the setting of Riemannian geometry and corresponding to the prestrained elastic materials. The effective 2d energy represents bending, relative to the prestrain tensor. We also take into account the incompressibility of the reference shell in view of the Riemannian metric.

1. INTRODUCTION

Shells are thin objects with one dimension very small compared to the other two and they naturally occur in a variety of scales: from thin graphene to petals of flowers to the atmospheric layer of the earth. Mathematically, a thin shell is represented by:

$$S^h = \left\{ z = x + t\vec{n}(x);\ x \in S,\ -\frac{h}{2} < t < \frac{h}{2} \right\},$$

where S is a 2d surface embedded in $\mathbb{R}^3$, $\vec{n}(x)$ is the unit normal vector to S at the point x, and $h > 0$ is the thickness of the shell. When $S \subset \mathbb{R}^2$, S^h is referred to as a plate.

In this paper, we will consider the asymptotic behavior of the energy functional:

$$I^h(u^h) = \frac{1}{h} \int_{S^h} W \left(\nabla u^h \left(\sqrt{g^h} \right)^{-1} \right), \tag{1.1}$$

defined on the set of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$. Here, g^h is a given Riemannian metric on S^h and $W : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}_+$ is a general elastic energy density. We assume that $I^h(u^h)$ is comparable to:

$$I^h_W(u^h) = \frac{1}{h} \int_{S^h} \mathrm{dist}^2 \left(\nabla u^h \left(\sqrt{g^h} \right)^{-1}, SO(3) \right), \tag{1.2}$$

and hence it measures how well the metric g^h is realized by the deformation u^h. Above, $SO(3)$ denotes the special orthogonal group in $\mathbb{R}^3$. The energy functional $I^h_W(u^h)$ arises from studies of shells with residual stress [6, 7, 12, 15], and the metric g^h describes the prescribed strain in the shell S^h.

In this paper, we adopt the Γ-convergence [4] from calculus of variations as the main approach. Such method has roots in classical plate/shell theory, where different scalings of the energy functional $I^h(u^h)$ in terms of h result in different 2d elastic theories [3, 8, 9, 13]. In classical plate/shell theory, Γ-convergence becomes a natural analytic tool, because it guarantees that the global minimizers of the 3d energy functionals converge to the global minimizers of the 2d limiting energy. Applying a similar approach, in [15] Lewicka and Pakzad studied pre-strained plates, i.e. $S \subset \mathbb{R}^2$ with metric g^h independent of thickness h, in the energy scaling regime given by $I^h(u^h) \le Ch^2$. The obtained result is a variant of a widely accepted theory in elasticity—Kirchhoff plate theory.

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Our paper first extends the discussion in [15] to shells and then includes a parallel discussion for the incompressible case. To be more specific, our first result is that for general surface S, with the same scaling of energy functionals and the same condition of metric g^h as in [15], the rescaled energy functional $h^{-2}I^h(u^h)$ Γ-converges, as the thickness $h \to 0$, to the functional:

$$\mathcal{I}(y) = \frac{1}{24}\int_S \mathcal{Q}_2\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\,\nabla\vec{N} - [g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right)\mathrm{d}x.$$

Here, the term $(\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\Pi$ measures the relative bending with respect to the desired metric. The detailed explanation of this fact and the exact meaning of each term in $\mathcal{I}(y)$ will be provided in the next section. Furthermore, taking into account the incompressibility of reference shell with respect to metric g^h, i.e. $\det\left(\nabla u^h(\sqrt{g^h})^{-1}\right) = 1$, we prove that the Γ-limit of rescaled energy functional $h^{-2}I^h(u^h)$ has the form:

$$\mathcal{I}_{In}(y) = \frac{1}{24}\int_S \mathcal{Q}_2^{In}\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right),$$

where:

$$\mathcal{Q}_2^{In}(x, F_{tan}) = \min_{d\in\mathbb{R}^3}\left\{\mathcal{Q}_3(F_{tan} + d\otimes\vec{n} + \vec{n}\otimes d); \mathrm{Tr}(F_{tan} + d\otimes\vec{n} + \vec{n}\otimes d) = 0\right\},$$

while in $\mathcal{I}(y)$:

$$\mathcal{Q}_2(x, F_{tan}) = \min\left\{\mathcal{Q}_3\left(\tilde{F}\right); \tilde{F}_{tan} = F_{tan}\right\}.$$

The organization of the paper is as follows: In section 2, an overview of the main results will be presented. Section 3 and 4 focus on the proof of the theory for shells. In section 5, we mainly discuss the proof for incompressible shells.

At last, it is also necessary to mention that Mahadevan, Lewicka and Pakzad applied a similar approach in [12] and rigourously justified the von Kármán-like equations for plates. These equations were first obtained via asymptotic expansion by Liang and Mahadevan [11].

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2. Overview of the Main Results

Consider a 2-dimensional surface S embedded in $\mathbb{R}^3$, which is compact, connected, oriented, and of class $\mathcal{C}^{1,1}$, and whose boundary ∂S is the union of finitely many (possibly none) Lipschitz continuous curves. For any $x \in S$, let $\vec{n}$ be the unit normal vector at x, T_xS the tangent space of S at x and $\Pi(x) = \nabla\vec{n}(x)$ the shape operator of S at x.

Assume a family of thin shells S^h around S have the following form:

$$S^h = \left\{z = x + t\vec{n}(x); x \in S, -\frac{h}{2} < t < \frac{h}{2}\right\}. \tag{2.1}$$

Let u^h be the deformation of S^h. Define the scaled elastic energy functional of u^h as:

$$I^h(u^h) = \frac{1}{h}\int_{S^h} W\left(\nabla u^h\left(\sqrt{g^h}\right)^{-1}\right), \tag{2.2}$$

where the energy density functional $W : \mathbb{R}^{3\times 3} \to \mathbb{R}_+$ satisfies frame indifference, normalization, growth and regularity as following:

$$\text{(HW)}\quad \left[\begin{array}{rl} \text{(i)} & W(RF) = W(F) \quad \forall F \in \mathbb{R}^{3\times 3}\ \forall R \in SO(3),\\ \text{(ii)} & W(\mathrm{Id}) = 0,\\ \text{(iii)} & W(F) \geq c\ \mathrm{dist}^2(F, SO(3)), \text{ with some constant } c > 0 \text{ independent of } F,\\ \text{(iv)} & W \text{ is } \mathcal{C}^2 \text{ in a neighborhood of the special orthogonal group } SO(3). \end{array}\right.$$

In accordance with [15], we assume that the metric

$$g^h(x + t\vec{n}) = g(x + t\vec{n}) = \begin{pmatrix} \begin{bmatrix} g_{\alpha\beta}(x) \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}, \ \forall x \in S,\ t \in \left(-\frac{h}{2}, \frac{h}{2}\right). \tag{2.3}$$

where $[g_{\alpha\beta}]$ is a smooth Riemann metric on S, defined up to the boundary. In particular, g does not depend on the thickness variable t.

The question we are interested in is the asymptotic behavior of rescaled energy functional $h^{-2}I^h(u^h)$ as the thickness h vanishes. Here, we investigate the situation when $I^h(u^h)$ scales like h^2. Thus, it is necessary for us to scale the shells S^h to a common domain S^{h_0}, where $0 < h_0 < 1$ is a positive number such that:

$$\frac{1}{2} < \det(\mathrm{Id} + t\Pi(x)) < \frac{3}{2}, \quad \frac{1}{2} < |\mathrm{Id} + t\Pi(x)| < \frac{3}{2} \quad \forall x \in S,\ \forall t \in \left(-\frac{h_0}{2}, \frac{h_0}{2}\right), \tag{2.4}$$

and so that the projection $\pi : S^{h_0} \longrightarrow S$ given by $\pi(z) = x$ is well defined for any $z = x + t\vec{n}(x) \in S^{h_0}$. Notice that when h is sufficiently small, such projection is also well defined for any $z \in S^h$. Now, we also adopt the following notation: For a 3×3 matrix field $M : S \longrightarrow \mathbb{R}^{3\times 3}$, we denote its tangential minor as M_{tan}, which is a bilinear form given by $M_{tan}(\tau, \eta) = \tau^T M \eta$, for each $\tau, \eta \in T_xS$. We shall also point out that when calculations in coordinate system are necessary, they are performed in some orthonormal basis $\{\tau_1(x), \tau_2(x), \vec{n}(x)\}$, where $\tau_1(x), \tau_2(x) \in T_xS$ and the basis satisfies the right-hand rule.

After introducing all these preliminaries, we are ready to state our first main result:

Theorem 2.1. (a) *(Compactness and lower bound) For a given sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ satisfying:*

$$I^h(u^h) \leq Ch^2, \tag{2.5}$$

where $C > 0$ is a uniform constant, there exists a sequence of translations $c^h \in \mathbb{R}^3$, such that the following properties hold for the normalized deformation:

$$\hat{y}^h(x + t\vec{n}) = u^h\left(x + t\frac{h}{h_0}\vec{n}(x)\right) - c^h, \quad \forall x \in S,\ \forall t \in \left(-\frac{h_0}{2}, \frac{h_0}{2}\right). \tag{2.6}$$

(i) *There exists $y \in W^{2,2}(S, \mathbb{R}^3)$ such that $y^h \to y \circ \pi$ in $W^{1,2}(S^{h_0}, \mathbb{R}^3)$ up to a subsequence.*

(ii) *Define $Q = \begin{bmatrix} \partial_{\tau_1} y & \partial_{\tau_2} y & \vec{N} \end{bmatrix}$, where $\vec{N} = \dfrac{\partial_{\tau_1} y \times \partial_{\tau_2} y}{|\partial_{\tau_1} y \times \partial_{\tau_2} y|}$, then $Q(\sqrt{g})^{-1} \in SO(3)$. Consequently, y realizes the midsurface metric i.e. $(\nabla y)^T(\nabla y) = [g_{\alpha\beta}]$ a.e. in S.*

(iii) *Define the following quadratic forms:*

$$\mathcal{Q}_3(F) = \nabla^2 W(\mathrm{Id})(F, F), \quad \mathcal{Q}_2(x, F_{tan}) = \min\left\{\mathcal{Q}_3\left(\tilde{F}\right); \tilde{F}_{tan} = F_{tan}\right\},$$

we then have the lower bound:

$$\liminf_{h\to 0} \frac{1}{h^2} I^h(u^h) \geq \mathcal{I}(y).$$

where:

$$\mathcal{I}(y)=\frac{1}{24}\int_S \mathcal{Q}_2\left(x,\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N}-[g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right)dx. \tag{2.7}$$

(b) *(Recovery Sequence) For every isometric immersion $y\in W^{2,2}(S,\mathbb{R}^3)$ of $[g_{\alpha\beta}]$, there exists a sequence of "recovery" deformations $u^h\in W^{1,2}(S^h,\mathbb{R}^3)$ such that the rescaled sequence $y^h(x+t\vec{n})=u^h(x+th/h_0\vec{n}(x))$ converges in $W^{1,2}(S^{h_0},\mathbb{R}^3)$ to y and*

$$\lim_{h\to 0}\frac{1}{h^2}I^h(u^h)=\mathcal{I}(y), \tag{2.8}$$

where $\mathcal{I}$ is defined as in (2.7).

Both $\mathcal{Q}_2(x,\cdot)$ and $\mathcal{Q}_3$ in Theorem 2.1 are quadratic forms, and $\mathcal{Q}_3$ is defined for $F\in\mathbb{R}^{3\times 3}$ while $\mathcal{Q}_2(x,\cdot)$ is defined on tangential minors F_{tan} of such matrices for a given $x\in S$. Note also that $\mathcal{Q}_3$ and $\mathcal{Q}_2(x,\cdot)$ are positive definite and depend only on the symmetric parts of their arguments [8]. Let $\tau,\eta\in T_xS$, and observe that the corresponding push-forward vectors by y is given by $\partial_\tau y$ and $\partial_\eta y$. Thus, we can calculate the relative bending as the difference between the action of the new bending function Π_{new} and the old bending function Π_{old} with respect to the desired metric $[g_{\alpha\beta}]$:

$$\begin{aligned}\Pi_{new}(\partial_\tau y,\partial_\eta y)-[g_{\alpha\beta}]\,\Pi_{old}(\tau,\eta)&=\langle\partial_\tau y,\Pi_{new}\partial_\eta y\rangle-\langle\tau,[g_{\alpha\beta}]\Pi\eta\rangle\\&=\left\langle\nabla y\tau,\nabla\vec{N}\eta\right\rangle-\langle\tau,[g_{\alpha\beta}]\,\Pi\eta\rangle=\left\langle\tau,\left[(\nabla y)^T\nabla\vec{N}-[g_{\alpha\beta}]\Pi\right]\eta\right\rangle.\end{aligned}$$

Thus, the term $(\nabla y)^T\nabla\vec{N}-g_{\alpha\beta}\Pi$ represents the relative bending and $\mathcal{I}$ represents the relative bending energy with the existence of the prestrained metric $[g_{\alpha\beta}]$.

We proceed to introduce the result for the incompressible analogue. With the presence of the desired metric g, being incompressible means shells perform deformations in accordance with the metric, i.e. $\det\nabla u^h=\det\sqrt{g}$ or $\det(\nabla u^h\left(\sqrt{g}\right)^{-1})=1$. The energy density functional $W_{In}:\mathbb{R}^{3\times 3}\longrightarrow[0,\infty]$ is defined as:

$$W_{In}(F)=\begin{cases}W(F), & \text{if } \det F=1,\\ \infty, & \text{otherwise,}\end{cases} \tag{2.9}$$

where W satisfies (HW), and the scaled energy functional is thus defined as:

$$I^h_{In}(u^h)=\frac{1}{h}\int_{S^h}W_{In}\left(\nabla u^h\left(\sqrt{g}\right)^{-1}\right). \tag{2.10}$$

With such settings, we can derive our second main result.

Theorem 2.2. *Assume furthermore that S is of class $\mathcal{C}^4$ and satisfies:*

(H) [*The set of $\mathcal{C}^3$ isometric immersions of $[g_{\alpha\beta}]$ on S is dense in the space of $W^{2,2}$ isometric immersions of $[g_{\alpha\beta}]$ with respect to the $W^{2,2}(S,\mathbb{R}^3)$ topology.*

Then Theorem 2.1 part (a) holds with I^h_{In} replacing I^h and the limiting energy functional

$$\mathcal{I}_{In}(y)=\frac{1}{24}\int_S\mathcal{Q}_2^{In}\left(x,\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N}-[g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right), \tag{2.11}$$

where:

$$\mathcal{Q}_2^{In}(x,F_{tan})=\min_{d\in\mathbb{R}^3}\{\mathcal{Q}_3(F_{tan}+d\otimes\vec{n}+\vec{n}\otimes d);\ Tr(F_{tan}+d\otimes\vec{n}+\vec{n}\otimes d)=0\}, \tag{2.12}$$

replacing $\mathcal{I}(y)$ defined in (2.7). With the same replacements Theorem 2.1 part (b) also holds.

Generally, it is difficult to characterize surface S and metric $[g_{\alpha\beta}]$ satisfying (H). Even when $[g_{\alpha\beta}]$ is a flat metric, this question has not been fully explored and the only results, valid for $S \subset \mathbb{R}^2$, are due to Pakzad and Hornung in [17] and [10] respectively. We conjecture that if $S \subset \mathbb{R}^2$ is convex and $[g_{\alpha\beta}]$ has strictly positive Gaussian curvature, condition (H) will be satisfied, with every $W^{2,2}$ isometric immersion being effectively smooth at the interior points of S.

Finally, let us formulate Theorem 2.1 in the language of Γ-convergence. Recall that a sequence of functionals $\mathcal{F}^h : X \longrightarrow \bar{\mathbb{R}}$ defined on a metric space X, is said to Γ-converge, as $h \to 0$, to $\mathcal{F} : X \longrightarrow \bar{\mathbb{R}}$ provided that the following two conditions hold:

(i) For any converging sequence $\{x^h\}$ in X:

$$\liminf_{h\to 0} \mathcal{F}^h(x^h) \geq \mathcal{F}\left(\lim_{h\to 0} x^h\right).$$

(ii) For every $x \in X$, there exists a sequence $\{x^h\}$ converging to x and such that:

$$\lim_{h\to 0} \mathcal{F}^h(x^h) = \mathcal{F}(x).$$

In view of this definition, Theorem 2.1 implies:

Corollary 2.3. *The sequence of functions $\mathcal{F}^h : W^{1,2}(S^{h_0}, \mathbb{R}^3) \longrightarrow \bar{\mathbb{R}}$, given by:*

$$\mathcal{F}^h(y^h) = \frac{1}{h^2} I^h(y^h),$$

Γ-converges, as $h \to 0$, to:

$$\mathcal{F}(y) = \begin{cases} \mathcal{I}(y), & \text{if } y \text{ is a } W^{2,2} \text{ isometric immersion of } [g_{\alpha\beta}], \\ +\infty, & \text{otherwise}. \end{cases} \tag{2.13}$$

Consequently, the (global) approximate minimizers of $\mathcal{F}^h$ converge to a global minimizer of $\mathcal{F}$.

3. Compactness and Lower Bound - Proof of Theorem 2.1 part (a)

The proof of Theorem 2.1 part (a) relies on the following lemma:

Lemma 3.1. *For each $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$, there exists a matrix field $Q^h \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ such that:*

$$\frac{1}{h}\int_{S^h} \left|\nabla u^h - Q^h \circ \pi\right|^2 \mathrm{d}z \leq C\left(h^2 + I^h(u^h)\right), \tag{3.1}$$

$$\int_S \left|\nabla Q^h\right|^2 \leq C\left(1 + h^{-2} I^h(u^h)\right), \tag{3.2}$$

with constant C independent of h.

The proof of Lemma 3.1 follows the technique developed in Theorem 10 [8] (See also Lemma 4.1 in [15]), and it is based on a equivalent version of the geometric rigidity estimate for Riemannian metrics proved in [15]:

Lemma 3.2. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open bounded Lipschitz domain with a given smooth Riemannian metric $g^* = [g^*_{ij}]$. Then, for every $u \in W^{1,2}(\mathcal{U}, \mathbb{R}^n)$ there exists $Q \in \mathbb{R}^{n\times n}$ such that:*

$$\int_{\mathcal{U}} |\nabla u(x) - Q|^2 \mathrm{d}x \leq C\left(\int_{\mathcal{U}} \mathrm{dist}^2(\nabla u\left(\sqrt{g^*(x)}\right)^{-1}, SO(3))\mathrm{d}x + \|\nabla g^*\|^2_{L^\infty} (\mathrm{diam}\ \mathcal{U})^2 |\mathcal{U}|\right),$$

where the constant C depends on g^* and on the domain $\mathcal{U}$. In particular, C is invariant under dilation of $\mathcal{U}$ and the dependence of C on $\mathcal{U}$ is uniform for a family of domains which are bilipschitz equivalent with controlled Lipschitz constants.

With Lemma 3.2, we may proceed to prove Lemma 3.1 as:

Proof of Lemma 3.1. For each $x \in S$, let $D_{x,h} = B(x,h) \cap S$ be 2d curvilinear disc in S of radius h and centered at a given x. Let $B_{x,h} = \pi^{-1}(D_{x,h})$ be the corresponding 3d cylinders. Applying Lemma 3.2 to each $B_{x,h}$ with metric g, we obtain that there exists $Q_{x,h}$ such that:

$$\begin{aligned}
&\int_{B_{x,h}} \left|\nabla u^h(z) - Q_{x,h}\right|^2 \mathrm{d}z \\
&\leq C\left(\int_{B_{x,h}} \mathrm{dist}^2\left(\nabla u\,(\sqrt{g})^{-1}, SO(3)\right)\mathrm{d}z + \|\nabla g\|^2_{L^\infty}(\mathrm{diam}B_{x,h})^2|B_{x,h}|\right) \\
&\leq C\int_{B_{x,h}} h^2 + \mathrm{dist}^2\left(\nabla u\,(\sqrt{g})^{-1}, SO(3)\right)\mathrm{d}z
\end{aligned} \tag{3.3}$$

where the constant C depends on the metric g and the region $B_{x,h}$.

Since S is of class $\mathcal{C}^{1,1}$ with boundary ∂S the union of finitely many (possibly none) Lipschitz continuous curves, then, for all $x \in S$, $B_{x,h}$ is bilipschitz equivalent to $B(0,h)$ with controlled Lipschitz constants. Besides the constant C is invariant under dilation, the constant C in (3.3) is uniform in x and h.

Consider now a sequence of functions $\eta_x : S^h \longrightarrow \mathbb{R}$ defined as:

$$\eta_x(z) = \frac{\vartheta(|\pi z - x|/h)}{\int_{S^h}\vartheta(|\pi y - x|/h)\mathrm{d}y}.$$

where $\vartheta \in \mathcal{C}^\infty_c([0,1))$ be a nonnegative cut-off function, equal to a positive constant in a neighborhood of 0.

Then $\eta_x(z) = 0$ for $z \notin B_{x,h}$ and:

$$\int_{S^h}\eta_x(z)\mathrm{d}z = 1, \qquad \|\eta_x\|_{L^\infty(S^h)} \leq Ch^{-3}, \qquad \|\nabla_x\eta_x\|_{L^\infty}(S^h) \leq Ch^{-4}. \tag{3.4}$$

Consider the approximation $Q^h \in W^{1,2}(S,\mathbb{R}^{3\times3})$ defined as $Q^h(x) = \int_{S^h}\eta_x(z)\nabla u^h(z)\mathrm{d}z$. By (3.3) and (3.4), for each $x \in S$, we have:

$$\begin{aligned}
\left|Q^h(x) - Q_{x,h}\right|^2 &= \left|\int_{S^h}\eta_x(z)\left(\nabla u^h(z) - Q_{x,h}\right)\mathrm{d}z\right|^2 \leq \int_{B_{x,h}}|\eta_x(z)|^2\int_{B_{x,h}}\left|\nabla u^h(z) - Q_{x,h}\right|^2\mathrm{d}z \\
&\leq Ch^{-3}\int_{B_{x,h}} h^2 + \mathrm{dist}^2\left(\nabla u^h\,(\sqrt{g})^{-1}, SO(3)\right)\mathrm{d}z,
\end{aligned}$$

and:

$$\begin{aligned}
\left|\nabla Q^h(x)\right|^2 &= \left|\int_{S^h}\nabla_x\eta_x(z)\left(\nabla u^h(z) - Q_{x,h}\right)\mathrm{d}z\right|^2 \leq \int_{B_{x,h}}|\nabla_x\eta_x(z)|^2\mathrm{d}z\int_{B_{x,h}}\left|\nabla u^h - Q_{x,h}\right|^2\mathrm{d}z \\
&\leq Ch^{-5}\int_{B_{x,h}} h^2 + \mathrm{dist}^2\left(\nabla u^h\,(\sqrt{g})^{-1}, SO(3)\right)\mathrm{d}z.
\end{aligned}$$

Similarly, for any $\tilde{x} \in D_{x,h}$, we have:

$$\begin{aligned}
\left|\nabla Q^h(\tilde{x})\right|^2 &\leq Ch^{-5}\int_{B_{\tilde{x},h}} h^2 + \mathrm{dist}^2\left(\nabla u^h\,(\sqrt{g})^{-1}, SO(3)\right)\mathrm{d}z \\
&\leq Ch^{-5}\int_{B_{x,2h}} h^2 + \mathrm{dist}^2\left(\nabla u^h\,(\sqrt{g})^{-1}, SO(3)\right)\mathrm{d}z.
\end{aligned}$$

Thus, in view of boundedness and connectedness of S, the fundamental theorem of calculus implies:

$$\left|Q^h(\tilde{x}) - Q^h(x)\right|^2 \le Ch^{-3} \int_{B_{x,2h}} h^2 + \mathrm{dist}^2 \left(\nabla u^h \left(\sqrt{g}\right)^{-1}, SO(3)\right) \mathrm{d}z, \quad \forall \tilde{x} \in D_{x,h}.$$

Therefore:

$$\begin{aligned}
&\int_{B_{x,h}} \left|\nabla u^h(z) - Q^h(\pi(z))\right|^2 \mathrm{d}z \\
&\le C\left(\int_{B_{x,h}} \left|\nabla u^h(z) - Q_{x,h}\right|^2 \mathrm{d}z + \int_{B_{x,h}} \left|Q_{x,h} - Q^h(x)\right|^2 \mathrm{d}z + \int_{B_{x',h}} \left|Q^h(x) - Q^h(\pi(z))\right|^2 \mathrm{d}z\right) \\
&\le C \int_{B_{x,2h}} h^2 + \mathrm{dist}^2 \left(\nabla u^h \left(\sqrt{g}\right)^{-1}, SO(3)\right) \mathrm{d}z,
\end{aligned}$$

and:

$$\int_{D_{x,h}} \left|\nabla Q^h\right|^2 \le Ch^{-3} \int_{B_{x,2h}} h^2 + \mathrm{dist}^2 \left(\nabla u^h \left(\sqrt{g}\right)^{-1}, SO(3)\right) \mathrm{d}z.$$

Now cover S by $\{D_{x_i,h}\}_{i=1}^{N_h}$, so that the covering number of the family $\{B_{x_i,h}\}_{i=1}^{N_h}$ is independent of h. One may refer to [14] for the existence of such covering. Summing the above inequalities over $i = 1, \cdots, N_h$, we obtain:

$$\int_{S^h} \left|\nabla u^h(z) - Q^h(\pi(z))\right|^2 \mathrm{d}z \le C \int_{S^h} h^2 + \mathrm{dist}^2 \left(\nabla u^h \left(\sqrt{g}\right)^{-1}, SO(3)\right) \mathrm{d}z, \tag{3.5}$$

$$\int_{S} \left|\nabla Q^h\right|^2 \le C + Ch^{-2} \frac{1}{h} \int_{S^h} \mathrm{dist}^2 \left(\nabla u^h \left(\sqrt{g}\right)^{-1}, SO(3)\right) \mathrm{d}z. \tag{3.6}$$

Besides $W(F) \ge c\,\mathrm{dist}^2(F, SO(3))$ implies:

$$\frac{1}{h} \int_{S^h} \mathrm{dist}^2 \left(\nabla u^h \left(\sqrt{g}\right)^{-1}, SO(3)\right) \mathrm{d}z \le CI^h(u^h). \tag{3.7}$$

Therefore, (3.5), (3.6) and (3.7) indicate (3.1) and (3.2). □

Proof of Theorem 2.1 part (a): 1. For $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$, applying Lemma 3.1, we obtain a sequence of matrix fields Q^h, which satisfy (3.1) and (3.2). Especially, notice that Q^h is bounded in $W^{1,2}(S, \mathbb{R}^{3\times3})$ in view of (2.5), (3.1) and (3.2). Together with the compact embedding theorem in Sobolev spaces, it implies that Q^h converges weakly in $W^{1,2}(S, \mathbb{R}^{3\times3})$ and strongly in $L^2(S, \mathbb{R}^{3\times3})$ to some matrix field $Q \in W^{1,2}(S, \mathbb{R}^{3\times3})$ up to a subsequence. At the meantime, denote the rescaled deformation $y^h(x + t\vec{n}) = u^h(x + th/h_0\vec{n})$ and $\nabla_h y^h(x + t\vec{n}) = \nabla u^h\,(x + th/h_0\vec{n})$. Note that:

$$\begin{aligned}
&\int_{S^{h_0}} \left|\nabla_h y^h - Q \circ \pi(z)\right|^2 \mathrm{d}z \\
&\le 2 \int_{S^{h_0}} \left|\nabla_h y^h(z) - Q^h \circ \pi(z)\right|^2 \mathrm{d}z + 2 \int_{S^{h_0}} \left|Q^h \circ \pi(z) - Q \circ \pi(z)\right|^2 \mathrm{d}z \\
&\le Ch^{-1} \int_{S^h} \left|\nabla u^h(z) - Q^h \circ \pi(z)\right|^2 \mathrm{d}z + C \int_S \left|Q^h(x) - Q(x)\right|^2 \mathrm{d}x.
\end{aligned}$$

Thus, in view of convergence of Q^h, (2.5) and (3.1), we obtain:

$$\nabla_h y^h \to Q \circ \pi \qquad \text{in } L^2(S^{h_0}) \text{ as } h \to 0.$$

On the other hand, we have:

$$\nabla y^h(x+t\vec{n}(x))\tau_i = \nabla_h y^h(x+t\vec{n}(x))\left(\mathrm{Id}+t\frac{h}{h_0}\Pi(x)\right)(\mathrm{Id}+t\Pi(x))^{-1}\tau_i, \quad \text{for } i=1,2;$$
$$\nabla y^h(x+t\vec{n}(x))\vec{n}(x) = \frac{h}{h_0}\nabla_h y^h(x+t\vec{n}(x))\vec{n}(x).$$

Hence, there follows:

$$\nabla y^h(x+t\vec{n}(x))\tau_i \to Q(x)(\mathrm{Id}+t\Pi(x))^{-1}\tau_i \quad \text{in } L^2(S^{h_0}) \quad \text{for } i=1,2;$$
$$\nabla y^h(x+t\vec{n}(x))\vec{n}(x) \to 0 \quad \text{in } L^2(S^{h_0}).$$

Set $c^h = \fint_{S^{h_0}} u^h\left(x+t\frac{h}{h_0}\vec{n}\right)\mathrm{d}x$, by means of Poincaré inequality and Sobolev embedding theory, the assertion (i) of the Theorem 2.1 part (a) follows. The higher regularity of y comes from the fact that $Q\in W^{1,2}(S,\mathbb{R}^{3\times 3})$. Also, we notice $\partial_\tau y = Q\tau$ for each $\tau\in T_xS$.

2. To prove (ii), notice that by (2.5), (3.1) and growth of W:

$$\begin{aligned}
&\int_S \mathrm{dist}^2\left(Q^h(\sqrt{g})^{-1}, SO(3)\right)\mathrm{d}x \le \frac{C}{h}\int_{S^h}\mathrm{dist}^2\left(Q^h(\sqrt{g})^{-1}, SO(3)\right)\\
&\le \frac{C}{h}\int_{S^h}\left(\mathrm{dist}^2\left(\nabla u^h(\sqrt{g})^{-1}, SO(3)\right)+\left|\nabla u^h - Q^h\circ\pi\right|^2\right)\mathrm{d}z \qquad (3.8)\\
&\le C\frac{1}{h}\int_{S^h} W(x,\nabla u(x))\mathrm{d}x + C\left(h^2+I^h(u^h)\right) = C\left(h^2+I^h(u^h)\right)\le Ch^2.
\end{aligned}$$

Hence, $Q(x)(\sqrt{g})^{-1}\in SO(3)$ a.e. in S and consequently $\partial_{\tau_i}y\cdot\partial_{\tau_j}y = [g_{\alpha\beta}]_{ij}$ for $i,j=1,2$, that is y realizes the midsurface metric $[g_{\alpha\beta}]$. Thus, we only need to show $Q\vec{n}=\vec{N}$, in order to do so, write $Q=R\sqrt{g}$ for some $R\in SO(3)$ and notice that:

$$\partial_{\tau_1}y\times\partial_{\tau_2}y = (Q\tau_1)\times(Q\tau_2) = R(\sqrt{g}\tau_1\times\sqrt{g}\tau_2).$$

Since $\sqrt{g} = \begin{pmatrix} \left[\sqrt{[g_{\alpha\beta}]}\right] & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & 1\end{pmatrix}$, $\sqrt{g}\tau_1\times\sqrt{g}\tau_2 = (\det\sqrt{g})\vec{n} = (\det\sqrt{g})\sqrt{g}\vec{n}$. Thus:

$$\partial_{\tau_1}y\times\partial_{\tau_2}y = (\det\sqrt{g})R\sqrt{g}\vec{n} = (\det\sqrt{g})Q\vec{n}.$$

Meanwhile $|\partial_{\tau_1}y\times\partial_{\tau_2}y| = |R(\sqrt{g}\tau_1\times\sqrt{g}\tau_2)| = |\sqrt{g}\tau_1\times\sqrt{g}\tau_2| = \det\sqrt{g}>0$. Then:

$$Q\vec{n}(x) = \frac{\partial_{\tau_1}y\times\partial_{\tau_2}y}{\det\sqrt{g}} = \frac{\partial_{\tau_1}y\times\partial_{\tau 2}y}{|\partial_{\tau_1}y\times\partial_{\tau 2}y|} = \vec{N}(x),$$

by which we finishes the proof of (ii).

3. We now modify the sequence Q^h to retain its convergence properties and additionally satisfies $\tilde{Q}^h(x)(\sqrt{g})^{-1}\in SO(3)$ for a.e. $x\in S$. We can define $\tilde{Q}^h(x)$ to be the matrix field on S such that:

$$\tilde{Q}^h(x)(\sqrt{g})^{-1} = \begin{cases} \pi_{SO(3)}\left(Q^h(x)\left(\sqrt{g}\right)^{-1}\right) & Q^h(x)\left(\sqrt{g}\right)^{-1}\in \text{small neighborhood of } SO(3),\\ \mathrm{Id} & \text{otherwise.}\end{cases}$$

where $\pi_{SO(3)}$ denotes the projection onto the compact set $SO(3)$ of its (sufficiently small) neighborhood. In view of (3.8) and the definition of projection, we see:

$$\int_S\left|\tilde{Q}^h - Q^h\right|^2 \le C\int_S\left|\tilde{Q}^h(\sqrt{g})^{-1} - Q^h(\sqrt{g})^{-1}\right|^2 \le C\int_S \mathrm{dist}^2\left(Q^h(x)(\sqrt{g})^{-1}, SO(3)\right)\mathrm{d}x \le Ch^2.$$

In particular, $\tilde{Q}^h$ converge to Q in $L^2(S)$.

Write $\tilde{Q}^h = R^h\sqrt{g}$ for a matrix field $R^h \in SO(3)$ and consider the rescaled strain:

$$G^h(x+t\vec{n}(x)) = \frac{1}{h}\left(\left(R^h(x)\right)^T \nabla_h y^h(x+t\vec{n}(x))\left(\sqrt{g(x)}\right)^{-1} - \mathrm{Id}\right). \tag{3.9}$$

Together with (2.5) and (3.1), such defined G^h is uniformly bounded in $L^2(S^{h_0},\mathbb{R}^{3\times 3})$, which implies that there exists $G \in L^2(S^{h_0},\mathbb{R}^{3\times 3})$, such that:

$$\lim_{h\to 0} G^h = G \text{ weakly in } L^2(S^{h_0},\mathbb{R}^{3\times 3}). \tag{3.10}$$

4. Fix a small $s>0$. Extend y^h periodically as following:

$$y^h(x+t\vec{n}) = \begin{cases} y^h(x+t\vec{n}), & \text{if } t\in(-h_0/2,h_0/2)\,, \\ y^h(x+(t-h_0)\vec{n}), & \text{if } t\in(h_0/2,h_0/2+s)\,. \end{cases} \tag{3.11}$$

Now consider the difference quotients:

$$f^{s,h}(x+t\vec{n}) = \frac{1}{h}\frac{1}{s}\left(y^h(x+(t+s)\vec{n}) - y^h(x+t\vec{n})\right) \in W^{1,2}(S^{h_0},\mathbb{R}^3).$$

Besides, $\partial_{\vec{n}} y^h = h/h_0 \partial_{\vec{n}} u^h\,(x+th/h_0\vec{n})$ gives us:

$$h^{-1}\partial_{\vec{n}} y^h \to h_0^{-1}\vec{N}\circ\pi \qquad \text{in } L^2(S^{h_0}).$$

The above convergence implies:

$$\lim_{h\to 0} f^{s,h}(x+t\vec{n}) = \lim_{h\to 0}\frac{1}{h}\fint_0^s \partial_{\vec{n}} y^h(x+(t+\tau)\vec{n})\mathrm{d}\tau = \frac{1}{h_0}\vec{N}(x) \quad \text{in } L^2(S^{h_0}).$$

Also, there follows the convergence of the normal derivatives, strongly in $L^2(S^{h_0})$:

$$\lim_{h\to 0}\partial_{\vec{n}} f^{s,h}(x+t\vec{n}) = \lim_{h\to 0}\frac{1}{s}\left(\frac{1}{h}\partial_{\vec{n}} y^h(x+(t+s)\vec{n}) - \frac{1}{h}\partial_{\vec{n}} y^h(x+t\vec{n})\right) = 0.$$

Meanwhile, concerning the tangential derivatives, for each $\tau\in T_xS$ we have:

$$\begin{aligned} &\nabla f^{s,h}(x+t\vec{n})(\mathrm{Id}+t\Pi)\tau \\ &\qquad = \frac{1}{h}\frac{1}{s}\left(\nabla y^h(x+(t+s)\vec{n})(\mathrm{Id}+(t+s)\Pi)\tau - \nabla y^h(x+t\vec{n})(\mathrm{Id}+t\Pi)\tau\right). \end{aligned} \tag{3.12}$$

On the other hand, in view of the definition of y^h,

$$\nabla y^h(x+t\vec{n})(\mathrm{Id}+t\Pi)\tau = \nabla_h y^h(x+t\vec{n})\left(\mathrm{Id}+t\frac{h}{h_0}\Pi\right)\tau. \tag{3.13}$$

Based on (3.9) and (3.13), we continue the calculation in (3.12) and obtain:

$$\begin{aligned} \nabla f^{s,h}(x+t\vec{n})(\mathrm{Id}+t\vec{n})\tau &= \frac{1}{h_0}\tilde{Q}^h(x)\Pi\tau + \frac{h}{h_0}R^h(x)G^h(x+(t+s)\vec{n})\sqrt{g(x)}\Pi\tau \\ &\quad + \tilde{Q}^h(x)\left(\sqrt{g(x)}\right)^{-1}\frac{1}{s}\left(G^h(x+(t+s)\vec{n}) - G^h(x+t\vec{n})\right)\sqrt{g(x)}\left(\mathrm{Id}+t\frac{h}{h_0}\Pi\right)\tau. \end{aligned} \tag{3.14}$$

By the weak convergence of G^h, boundedness of $\sqrt{g}$ and $(\sqrt{g})^{-1}$ and the convergence of $\tilde{Q}^h$, there follows that in $L^2(S^{h_0})$:

$$\nabla f^{s,h}(x+t\vec{n})(\mathrm{Id}+t\Pi)\tau \rightharpoonup \frac{1}{h_0}Q\Pi\tau + \frac{1}{s}Q(\sqrt{g})^{-1}\left(G(x+(t+s)\vec{n}) - G(x+t\vec{n})\right)\sqrt{g}\tau. \tag{3.15}$$

Consequently, the sequence $f^{s,h}$ converges, as $h \to 0$ weakly in $W^{1,2}(S^{h_0})$ to $h_0^{-1}\vec{N} \circ \pi$. Equating the tangential derivatives, it yields:

$$\frac{1}{h_0}\nabla\vec{N}\tau = \frac{1}{h_0}Q\Pi\tau + \frac{1}{s}Q(\sqrt{g})^{-1}\left(G(x+(t+s)\vec{n}) - G(x+t\vec{n})\right)\sqrt{g}\tau.$$

Therefore,

$$G(x+t\vec{n})\tau = G(x)\tau + \frac{t}{h_0}(\sqrt{g})Q^{-1}(\nabla\vec{N} - Q\nabla\vec{n})(\sqrt{g})^{-1}\tau. \tag{3.16}$$

5. We now calculate the lower bound of the rescaled energies. To this end, define:

$$S_h^0 = \left\{x + t\vec{n} \in S^{h_0}; \left|G^h(x+t\vec{n})\right| \leq h^{-1/2}\right\},$$

and their characteristic functions $\chi_h = \chi_{S_h^0}$, which by (3.10) converge in $L^1(S^{h_0})$ to 1.

Notice that in S_h^0, based on the properties of W, there follows the formal Taylor expansion:

$$\frac{1}{h^2}W\left(\nabla_h y^h(x+t\vec{n})(\sqrt{g})^{-1}\right) = \frac{1}{h^2}W\left(\mathrm{Id} + hG^h(x+t\vec{n})\right) = \frac{1}{2}\mathcal{Q}_3(G^h) + o(1)|G^h|^2,$$

where $o(1)$ is the Landau symbol denoting any quantity uniformly converging to 0, as $h \to 0$. Thus, we obtain:

$$\begin{aligned}
\frac{1}{h^2}I^h(u^h) &= \frac{1}{h^2}\int_S \fint_{-h_0/2}^{h_0/2} W\left(\nabla_h y^h(x+t\vec{n})\left(\sqrt{g(x)}\right)^{-1}\right)\det\left(\mathrm{Id} + t\frac{h}{h_0}\Pi\right)\mathrm{d}t\mathrm{d}x \\
&\geq \frac{1}{h^2}\int_S \fint_{-h_0/2}^{h_0/2} \chi_h(x+t\vec{n})W\left(\nabla_h y^h(x+t\vec{n})\left(\sqrt{g(x)}\right)^{-1}\right)\det\left(\mathrm{Id} + t\frac{h}{h_0}\Pi\right)\mathrm{d}t\mathrm{d}x \\
&= \int_S \fint_{-h_0/2}^{h_0/2} \chi_h(x+t\vec{n})\left[\frac{1}{2}\mathcal{Q}_3\left(G^h(x+t\vec{n})\right) + o(1)\left|G^h(x+t\vec{n})\right|^2\right]\det\left(\mathrm{Id} + t\frac{h}{h_0}\Pi\right)\mathrm{d}t\mathrm{d}x.
\end{aligned}$$

Hence:

$$\begin{aligned}
&\liminf_{h\to 0}\frac{1}{h^2}I^h(u^h) \\
&\geq \frac{1}{2}\liminf_{h\to 0}\int_S \fint_{-h_0/2}^{h_0/2} \chi_h(x+t\vec{n})\mathcal{Q}_3\left(G^h(x+t\vec{n})\right)\det\left(\mathrm{Id} + t\frac{h}{h_0}\Pi\right)\mathrm{d}t\mathrm{d}x \\
&= \frac{1}{2}\int_S \fint_{h_0/2}^{h_0/2} \mathcal{Q}_3\left(G(x+t\vec{n})\right)\mathrm{d}t\mathrm{d}x \geq \frac{1}{2}\int_S \fint_{-h_0/2}^{h_0/2} \mathcal{Q}_2\left(x, (G(x+t\vec{n}))_{tan}\right)\mathrm{d}t\mathrm{d}x.
\end{aligned} \tag{3.17}$$

Above, we used the fact that $\chi_h G^h$ converges weakly in $L^2(S^{h_0}, \mathbb{R}^{3\times 3})$ to G, derived from the convergent properties and boundedness of G^h and χ_h, and the nonnegative definiteness of the quadratic forms $\mathcal{Q}_3$. By (3.16), we have:

$$\begin{aligned}
\mathcal{Q}_2\left(x, G(x+t\vec{n})_{tan}\right) &= \frac{t^2}{h_0^2}\mathcal{Q}_2\left(x, \left(\sqrt{g}Q^{-1}\left(\nabla\vec{N} - Q\nabla\vec{n}\right)(\sqrt{g})^{-1}\right)_{tan}\right) \\
&+ \frac{2t}{h_0}\mathcal{L}_2(x)\left((G(x))_{tan}, \left(\sqrt{g}Q^{-1}\left(\nabla\vec{N} - Q\nabla\vec{n}\right)(\sqrt{g})^{-1}\right)_{tan}\right) + \mathcal{Q}_2\left(x, (G(x))_{tan}\right),
\end{aligned} \tag{3.18}$$

where $\mathcal{L}_2(x)$ is the correspondent bilinear form of $\mathcal{Q}_2(x, \cdot)$. Substituting (3.18) into (3.17), after dropping the first nonnegative term, there follows:

$$\liminf_{h\to 0}\frac{1}{h^2}I^h(u^h) \geq \frac{1}{24}\int_S \mathcal{Q}_2\left(x, \left(\sqrt{g}Q^{-1}\left(\nabla\vec{N} - Q\Pi\right)(\sqrt{g})^{-1}\right)_{tan}\right).$$

Since $Q(\sqrt{g})^{-1} \in SO(3)$, $\sqrt{g}Q^{-1} = (Q(\sqrt{g})^{-1})^T = (\sqrt{g})^{-1}Q^T$. Thus:

$$(\sqrt{g}Q^{-1}(\nabla\vec{N} - Q\nabla\vec{n})(\sqrt{g})^{-1})_{tan} = \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1},$$

which ends the proof of Theorem 2.1 part (a).

4. The Recovery Sequence - A Proof of Theorem 2.1 part (b)

In this section, we will construct a suitable recovery sequence y^h. Based on the definition and properties of $\mathcal{Q}_2(x,\cdot)$ and $\mathcal{Q}_3$, with a little abuse of notation, we may denote:

$$\mathcal{Q}_2(x, F_{tan}) = \min\{\mathcal{Q}_3(F_{tan} + c \otimes \vec{n}(x)); c \in \mathbb{R}^3\}. \tag{4.1}$$

In view of properties of $\mathcal{Q}_3$, it has a unique minimizer $c(x, F_{tan})$, which realizes $\mathcal{Q}_2(x, F_{tan})$. The uniqueness implies that c is linear in its second argument.

The proof of Theorem 2.1 part (b): 1. By the argument similar to the one we used in the proof of (ii) in Theorem 2.1 part (a), first notice that the matrix field Q defined as:

$$Q(x) = \begin{bmatrix}\partial_{\tau_1}y(x) & \partial_{\tau_2}y(x) & \vec{N}(x)\end{bmatrix} \quad \text{with } \vec{N}(x) = \frac{\partial_{\tau_1}y(x) \times \partial_{\tau_2}y(x)}{|\partial_{\tau_1}y(x) \times \partial_{\tau_2}y(x)|},$$

satisfies $Q(\sqrt{g})^{-1} \in SO(3)$. In view of the definition of c, we put:

$$d(x) = Q(x)\left(\sqrt{g(x)}\right)^{-1} \cdot c\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right) \in L^2(S, \mathbb{R}^3). \tag{4.2}$$

Let $d^h \in W^{1,\infty}(S, \mathbb{R}^3)$ be such that:

$$\lim_{h\to 0} d^h = d \quad \text{in } L^2(S) \quad \text{and} \quad \lim_{h\to 0} h\|d^h\|_{W^{1,\infty}(S,\mathbb{R}^3)} = 0. \tag{4.3}$$

Note that a sequence d^h with properties (4.3) can always be derived by reparametrizing (slowing down) a sequence of smooth approximations of the given vector field $d \in L^2(S)$.

We now define:

$$u^h(x + t\vec{n}) = \tilde{y}^h(x) + t\vec{N}^h(x) + \frac{1}{2}t^2 d^h(x), \tag{4.4}$$

where $\tilde{y}^h \in W^{2,\infty}(S, \mathbb{R}^3)$ and $\vec{N}^h \in W^{1,\infty}(S, \mathbb{R}^3)$ are approximations of y and $\vec{N}$ respectively and they enjoy the following properties:

$$\begin{aligned}
&\lim_{h\to 0}\|\tilde{y}^h - y\|_{W^{2,2}(S)} = 0, \qquad \lim_{h\to 0}\|\vec{N}^h - \vec{N}\|_{W^{1,2}(S)} = 0,\\
&h\left(\|\tilde{y}^h\|_{W^{2,\infty}(S)} + \|\vec{N}^h\|_{W^{1,\infty}(S)}\right) \le \varepsilon_0,\\
&\lim_{h\to 0}\frac{1}{h^2}\left|\left\{x \in S; \tilde{y}^h(x) \ne y(x)\right\} \cup \left\{x \in S; \vec{N}^h(x') \ne \vec{N}(x)\right\}\right| = 0,
\end{aligned} \tag{4.5}$$

for a sufficiently small, fixed number $\varepsilon_0 > 0$, to be chosen later. The existence of such approximation follows by partition of unity and a truncation argument, as a special case of the Lusin-type result for Sobolev functions in [16] (see also Proposition 2 in [9]).

Then, notice that the scaled deformations y^h have the following form:

$$y^h(x + t\vec{n}(x)) = u^h\left(x + t\frac{h}{h_0}\vec{n}(x)\right) = \tilde{y}^h(x) + t\frac{h}{h_0}\vec{N}^h(x) + t^2\frac{h^2}{2h_0^2}d^h(x). \tag{4.6}$$

By direct calculation, we can prove $y^h \to y \circ \pi$ in $W^{1,2}(S^{h_0}, \mathbb{R}^3)$.

2. We will devote the following to estimate the limit of the scaled energy functional. The key point is to calculate $\nabla u^h(x+th/h_0\vec{n})$ and we can find it through computation of tangential derivative and normal derivative respectively. Notice that for each $\tau \in T_xS$,

$$\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)\tau = \partial_\tau \tilde{y}^h + t\frac{h}{h_0}\left(\nabla \vec{N}^h - \nabla \tilde{y}^h \nabla \vec{n}\right)\tau + o(h), \tag{4.7}$$

where $o(h)$ satisfies:

$$\lim_{h\to 0}\frac{o(h)}{h} = 0 \quad \text{in } L^2(S), \tag{4.8}$$

while concerning the normal derivative, we have:

$$\partial_{\vec{n}} u^h\left(x+t\frac{h}{h_0}\vec{n}\right) = \vec{N}^h(x) + t\frac{h}{h_0}d^h(x). \tag{4.9}$$

Thus,

$$\begin{aligned}\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right) &= \left[\partial_{\tau_1}\tilde{y}^h \quad \partial_{\tau_2}\tilde{y}^h \quad \vec{N}^h\right] \\ &\quad + t\frac{h}{h_0}\left[\partial_{\tau_1}\vec{N}^h - \nabla\tilde{y}^h\partial_{\tau_1}\vec{n} \quad \partial_{\tau_2}\vec{N}^h - \nabla\tilde{y}^h\partial_{\tau_2}\vec{n} \quad d^h\right] + o(h).\end{aligned} \tag{4.10}$$

Denote $Q^h(x) = \left(\partial_{\tau_1}\tilde{y}^h \quad \partial_{\tau_2}\tilde{y}^h \quad \vec{N}^h\right)$ and $B_h(x) = \left(\partial_{\tau_1}\vec{N}^h - \nabla\tilde{y}^h\partial_{\tau_1}\vec{n} \quad \partial_{\tau_2}\vec{N}^h - \nabla\tilde{y}^h\partial_{\tau_2}\vec{n} \quad d^h\right)$. Let $S_h = \{x \in S; \tilde{y}^h(x) = y(x) \text{ and } \vec{N}^h(x) = \vec{N}(x)\}$. Performing a similar argument mentioned in [15], we attain that each map: $S \ni x' \mapsto \mathrm{dist}\{Q^h(x)(\sqrt{g})^{-1}, SO(3)\}$ vanishes on S_h and is Lipschitz in S, with Lipschitz constant of order $\mathcal{O}(1/h)$.

For any point $x \in S\setminus S_h$, there follows $\mathrm{dist}^2(x, S_h) \le C|S\setminus S_h|$. Concerning the detailed proof of the statement, one can refer to Lemma 6.1 in [12]. As a consequence, by (4.5), we obtain $1/h^2\mathrm{dist}^2(x,S_h)\to 0$ and hence:

$$\mathrm{dist}\left(Q^h(x)(\sqrt{g})^{-1}, SO(3)\right) \le \mathcal{O}\left(\frac{1}{h}\right)\mathrm{dist}(x,S_h) = o(1). \tag{4.11}$$

Hence, in view of the boundedness of $\tilde{y}^h$, $\vec{N}^h$ and d^h, when h and ε_0 are sufficiently small, $\nabla u^h(x+th/h_0\vec{n})(\sqrt{g})^{-1}$ is in the neighborhood of $SO(3)$, and thus $W(\nabla u^h(x+th/h_0\vec{n})(\sqrt{g})^{-1})$ is also bounded. We proceed to consider the scaled energy functional:

$$\begin{aligned} I^h(u^h) &= \frac{1}{h}\int_{S^h} W(\nabla u^h)\mathrm{d}z \\ &= \int_S \fint_{-h_0/2}^{h_0/2} W\left(\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)\left(\sqrt{g(x)}\right)^{-1}\right)\det\left(\mathrm{Id}+t\frac{h}{h_0}\nabla\vec{n}\right)\mathrm{d}t\mathrm{d}x.\end{aligned} \tag{4.12}$$

From the definition of S_h, (4.5), boundedness of W and the regularity of S, there follows:

$$\lim_{h\to 0}\frac{1}{h^2}\int_{S\setminus S_h}\fint_{-h_0/2}^{h_0/2} W\left(\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)\left(\sqrt{g(x)}\right)^{-1}\right)\det\left(\mathrm{Id}+t\frac{h}{h_0}\nabla\vec{n}\right)\mathrm{d}t\mathrm{d}x = 0$$

Thus, we shall focus our discussion on:

$$\frac{1}{h^2}\int_{S_h}\fint_{-h_0/2}^{h_0/2} W\left(\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)\left(\sqrt{g(x)}\right)^{-1}\right)\det\left(\mathrm{Id}+t\frac{h}{h_0}\nabla\vec{n}\right)\mathrm{d}t\mathrm{d}x.$$

Furthermore, the first step is to investigate the energy density functional W evaluated at $\nabla u^h(x+th/h_0\vec{n})(\sqrt{g})^{-1}$ for $x+t\vec{n} \in S_h^1 = \{z = x+t\vec{n}; x \in S_h, t \in (-h_0/2, h_0/2)\}$. Since $Q(\sqrt{g})^{-1} \in SO(3)$, by frame indifference and the formal Taylor expansion of W, there follows:

$$\begin{aligned}
&W\left(\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)(\sqrt{g})^{-1}\right)\\
&= W\left(Q(\sqrt{g})^{-1} + t\frac{h}{h_0}B_h(x)(\sqrt{g})^{-1} + o(h)\right) = W\left(\mathrm{Id} + t\frac{h}{h_0}(\sqrt{g})^{-1}Q^T B_h(x)(\sqrt{g})^{-1} + o(h)\right)\\
&= \frac{1}{2}h^2\mathcal{Q}_3\left(\frac{t}{h_0}(\sqrt{g})^{-1}Q^T B_h(x)(\sqrt{g})^{-1} + \frac{o(h)}{h}\right) + o(1)h^2\left|\frac{t}{h_0}(\sqrt{g})^{-1}Q^T B_h(x)(\sqrt{g})^{-1} + \frac{o(h)}{h}\right|^2.
\end{aligned}$$

Thus:

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h^2}I^h(u^h) &= \lim_{h\to 0}\frac{1}{h^2}\int_{S_h}\fint_{-h_0/2}^{h_0/2} W\left(\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)(\sqrt{g})^{-1}\right)\det\left(\mathrm{Id}+t\frac{h}{h_0}\Pi\right)\mathrm{d}t\mathrm{d}x\\
&= \lim_{h\to 0}\int_{S_h}\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_3\left(\frac{t}{h_0}(\sqrt{g})^{-1}Q^T B_h(\sqrt{g})^{-1} + \frac{o(h)}{h}\right)\det\left(\mathrm{Id}+t\frac{h}{h_0}\nabla\vec{n}\right)\mathrm{d}t\mathrm{d}x\\
&= \frac{1}{24}\int_S \mathcal{Q}_3\left(\left(\sqrt{g(x)}\right)^{-1}Q(x)^T B(x)\left(\sqrt{g(x)}\right)^{-1}\right)\mathrm{d}x,
\end{aligned}$$

where:

$$B(x) = \left[(\nabla\vec{N} - \nabla y\Pi)\tau_1 \quad (\nabla\vec{N} - \nabla y\Pi)\tau_2 \quad d\right].$$

Nevertheless, direct calculation shows that:

$$\begin{aligned}
\left(\sqrt{g(x)}\right)^{-1}Q(x)^T B(x)\left(\sqrt{g(x)}\right)^{-1} &= \left(\sqrt{g_{\alpha\beta}}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - g_{\alpha\beta}\Pi\right)\left(\sqrt{g_{\alpha\beta}}\right)^{-1}\\
&\quad + c\left(x, \left(\sqrt{g_{\alpha\beta}}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - g_{\alpha\beta}\nabla\vec{n}\right)\left(\sqrt{g_{\alpha\beta}}\right)^{-1}\right)\otimes\vec{n}.
\end{aligned}$$

Hence:

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h^2}I^h(u^h) &= \frac{1}{24}\int_S \mathcal{Q}_3\left(\left(\sqrt{g(x)}\right)^{-1}Q(x)^T B(x)\left(\sqrt{g(x)}\right)^{-1}\right)\mathrm{d}x\\
&= \frac{1}{24}\int_S \mathcal{Q}_2\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right)\mathrm{d}x.
\end{aligned}$$

5. Prestrained incompressible Shells–Proof of Theorem 2.2

In view of Theorem 2.1 and features of shells, we adopt the method developed in [2] in the proof.

Proof of Theorem 2.2 Part (a): The compactness and the lower bound follow from a combination of the corresponding results in Theorem 2.1 and the technique developed for the incompressible Kirchhoff model in [2].

Define, for all $k > 0$:

$$W_k(F) = W_c(F) + \frac{k}{2}(\det F - 1)^2,$$

and:

$$I_k^h(u^h) = \frac{1}{h}\int_{S^h} W_k\left(\nabla u^h(\sqrt{g})^{-1}\right)\,\mathrm{d}z.$$

As in [2], consider the quadratic forms associated with W_k:

$$\mathcal{Q}_3^k(F) = \nabla^2 W_k(\mathrm{Id})(F,F) = \mathcal{Q}_3(F) + k(\mathrm{Tr}F)^2,$$

and the corresponding reduced forms:

$$\mathcal{Q}_2^k(x, F_{tan}) = \min_{d\in\mathbb{R}^3} \mathcal{Q}_3^k(F_{tan} + d \otimes \vec{n} + \vec{n} \otimes d), \qquad \forall F_{tan} \in \mathbb{R}^{2\times 2}.$$

Applying the same argument as in Lemma 2.1 in [2], we obtain for each $F_{tan} \in \mathbb{R}^{2\times 2}$, $x \in S$, $k > 0$:

$$\mathcal{Q}_2^{In}(x, F_{tan}) - \frac{C}{\sqrt{k}}\|F_{tan}\|^2 \le \mathcal{Q}_2^k(x, F_{tan}) \le \mathcal{Q}_2^{In}(x, F_{tan}), \tag{5.1}$$

where the constant C is independent of k.

Since $I_{In}^h(u^h) \le Ch^2$, especially it is bounded, $W(\nabla u^h(\sqrt{g})^{-1})$ is a.e. finite, which implies $\det(\nabla u^h(\sqrt{g})^{-1}) = 1$ for a.e. $z \in S^h$. Thus, $W^k(\nabla u^h(\sqrt{g})^{-1}) = W(\nabla u^h(\sqrt{g})^{-1})$ a.e.. Hence, for a fixed k, $I_k^h(u^h) \le I^h(u^h) \le Ch^2$. From Theorem 2.1, (i) and (ii) follow and for each k:

$$\begin{aligned}\liminf_{h\to 0} \frac{1}{h^2} I^h(u^h) &\ge \liminf_{h\to 0} \frac{1}{h^2} I_h^k(u^h) \\ &\ge \frac{1}{24}\int_S \mathcal{Q}_2^k\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right).\end{aligned} \tag{5.2}$$

Notice that:

$$\begin{aligned}&\mathcal{Q}_2^k\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right) \\ &\quad\ge \mathcal{Q}_2^{In}\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right) \\ &\qquad - \frac{C}{\sqrt{k}}\left\|\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right\|^2.\end{aligned} \tag{5.3}$$

On the other hand, the nonnegative definiteness of $\mathcal{Q}_2^k(x,\cdot)$, the bound for energy $I^h(u^h)$ and(5.2) indicate:

$$\int_S \left\|\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right\|^2 \le C.$$

Passing the limit $k \to +\infty$, there follows:

$$\begin{aligned}&\lim_{k\to+\infty}\int_S \mathcal{Q}_2^k\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right) \\ &\qquad\ge \int_S \mathcal{Q}_2^{In}\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right).\end{aligned}$$

Therefore, (iii) follows and this complete the proof of part (a).

Proof of Theorem 2.2 part (b): A core assumption for this part: $\mathcal{C}^3(S;\mathbb{R}^3)$ isometries are dense in $W^{2,2}(S;\mathbb{R}^3)$ isometries. With this assumption, part (b) will follow through the diagonal argument and the application of the next two Lemmas.

Lemma 5.1. *Let $y \in \mathcal{C}^3(S,\mathbb{R}^3)$ with the unit normal to the deformed surface $\vec{N} = \dfrac{\partial_{\tau_1}y \times \partial_{\tau_2}y}{|\partial_{\tau_1}y \times \partial_{\tau_2}y|}$ satisfy that $Q(\sqrt{g})^{-1} = \begin{bmatrix}\partial_{\tau_1}y & \partial_{\tau_2}y & \vec{N}\end{bmatrix}(\sqrt{g})^{-1} \in SO(3)$. Assume $d \in \mathcal{C}^1(S,\mathbb{R}^3)$ enjoys the* *property:*

$$\mathrm{Tr}\left(\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N}-[g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}+Q^T d\otimes\vec{n}\right)=0.$$

Then there exists a sequence of deformations $u^h\in\mathcal{C}^1(S^h,\mathbb{R}^3)$ such that:

(i) *The rescaled sequence $y^h(x+t\vec{n}(x))=u^h(x+th/h_0\vec{n}(x))\to y(x)$ in $W^{1,2}(S^{h_0})$.*

(ii) *The upper limit of the scaled energy functional:*

$$\limsup_{h\to 0}\frac{1}{h^2}I^h(u^h)\le\frac{1}{24}\int_S\mathcal{Q}_3\left(\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N}-[g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}+Q^T d\otimes\vec{n}\right)dx.$$

Proof. Similar to the situation in Theorem 2.1, without taking into account of incompressibility of S^h, define the rescaled deformation $y_c^h\in\mathcal{C}^1(S^{2h_0},\mathbb{R}^3)$ as:

$$y_c^h(x+t\vec{n})=y(x)+t\frac{h}{h_0}\vec{N}(x)+t^2\frac{h^2}{2h_0^2}d(x).$$

Thus, we may define the corresponding deformation without rescaling as $u_c^h(x+t\vec{n})=y_c^h(x+th_0/h\vec{n})$. At the meantime, by chain rule we have:

$$\partial_\tau y_c^h(x+t\vec{n})=\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)\left(\mathrm{Id}+t\frac{h}{h_0}\nabla\vec{n}\right)(\mathrm{Id}+t\Pi)^{-1}\tau,\quad\forall\tau\in T_xS,$$
$$\partial_{\vec{n}}y_c^h(x+t\vec{n})=\frac{h}{h_0}\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right).$$

Thus, for each $\tau\in T_xS$:

$$\begin{aligned}\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)\tau&=\nabla y_c^h(x+t\vec{n})(\mathrm{Id}+t\Pi)\left(\mathrm{Id}+t\frac{h}{h_0}\Pi\right)^{-1}\tau\\&=\left(\nabla y+t\frac{h}{h_0}\nabla\vec{N}+t^2\frac{h^2}{2h_0^2}\nabla d\right)\left(\mathrm{Id}-t\frac{h}{h_0}\Pi+o(h)\right)\tau\\&=\partial_\tau y+t\frac{h}{h_0}\left(\partial_\tau\vec{N}-\nabla y\partial_\tau\vec{n}\right)+o(h),\end{aligned}$$

and:

$$\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)\vec{n}=\frac{h_0}{h}\partial_{\vec{n}}y_c^h(x+t\vec{n})=\vec{N}(x)+t\frac{h}{h_0}d(x).$$

Hence:

$$\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)=Q(x)+t\frac{h}{h_0}\left[\partial_{\tau_1}\vec{N}-\nabla y\partial_{\tau_1}\vec{n}\quad\partial_{\tau_2}\vec{N}-\nabla y\partial_{\tau_2}\vec{n}\quad d\right]+o(h).$$

Furthermore:

$$\begin{aligned}&\det\left(\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)(\sqrt{g})^{-1}\right)=\det\left(\left[Q\,(\sqrt{g})^{-1}\right]^T\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)(\sqrt{g})^{-1}\right)\\&=\det\left(\mathrm{Id}+t\frac{h}{h_0}(\sqrt{g})^{-1}Q^T\left[\partial_{\tau_1}\vec{N}-\nabla y\partial_{\tau_1}\vec{n}\quad\partial_{\tau_2}\vec{N}-\nabla y\partial_{\tau_2}\vec{n}\quad d\right](\sqrt{g})^{-1}+o(h)\right)\\&=1+t\frac{h}{h_0}\mathrm{Tr}\left(\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N}-[g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}+Q^T d\otimes\vec{n}\right)+o(h)=1+o(h).\end{aligned}$$

Thus,

$$\left|\det\left(\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)(\sqrt{g})^{-1}\right)-1\right|+\left|\partial_\tau\det\left(\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)(\sqrt{g})^{-1}\right)\right|=o(h).$$

We shall now consider the restriction of incompressibility and find the rescaled recovery sequence of deformation:

$$y^h(x+t\vec{n})=y_c^h\left(x+\phi^h(x,t)\vec{n}\right),$$

or equivalently:

$$u^h(x+s\vec{n})=u_c^h\left(x+\frac{h}{h_0}\phi^h\left(x,s\frac{h_0}{h}\right)\vec{n}\right),$$

for some function $\phi^h : S\times(-h_0/2,h_0/2)\longrightarrow\mathbb{R}$ with $\phi^h(x,0)=0$ for each $x\in S$. Then, for every $\tau\in T_xS$ we have:

$$\partial_\tau u^h(x+s\vec{n})=\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h\left(x,s\frac{h_0}{h}\right)\vec{n}\right)\left(\mathrm{Id}+\frac{h}{h_0}\nabla\left(\phi^h(x,s\frac{h_0}{h})\vec{n}(x)\right)\right)(\mathrm{Id}+s\Pi)^{-1}\tau.$$

Meanwhile:

$$\partial_\tau\left(\phi^h\left(x,s\frac{h_0}{h}\right)\vec{n}\right)=\phi^h(x,s\frac{h_0}{h})\partial_\tau\vec{n}+\partial_\tau\phi^h\left(x,s\frac{h_0}{h}\right)\vec{n}.$$

Therefore:

$$\begin{aligned}&\partial_\tau u(x+s\vec{n})\\&=\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h\left(x,\frac{h_0}{h}s\right)\vec{n}\right)\begin{bmatrix}\mathrm{Id}+\frac{h}{h_0}\phi^h\left(x,\frac{h_0}{h}s\right)\Pi\\ \frac{h}{h_0}\partial_{\tau_1}\phi^h\left(x,\frac{h_0}{h}s\right),\ \frac{h}{h_0}\partial_{\tau_2}\phi^h\left(x,\frac{h_0}{h}s\right)\end{bmatrix}(\mathrm{Id}+s\Pi)^{-1}\tau\\&=\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h\left(x,\frac{h_0}{h}s\right)\vec{n}\right)\begin{bmatrix}\left(\mathrm{Id}+\frac{h}{h_0}\phi^h\left(x,\frac{h_0}{h}s\right)\Pi\right)(\mathrm{Id}+s\Pi)^{-1}\\ \left[\frac{h}{h_0}\partial_{\tau_1}\phi^h\left(x,\frac{h_0}{h}s\right)\quad\frac{h}{h_0}\partial_{\tau_2}\phi^h\left(x,\frac{h_0}{h}s\right)\right](\mathrm{Id}+s\Pi)^{-1}\end{bmatrix}\tau.\end{aligned}$$

For the normal derivative:

$$\partial_{\vec{n}}u^h(x+s\vec{n})=\partial_t\phi^h\left(x,\frac{h_0}{h}s\right)\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h\left(x,\frac{h_0}{h}s\right)\vec{n}(x)\right)\vec{n}.$$

Consequently:

$$\nabla u^h(x+s\vec{n})=\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h\left(x,\frac{h_0}{h}s\right)\vec{n}(x)\right)M_1,$$

where:

$$M_1=\begin{bmatrix}\left(\mathrm{Id}+\frac{h}{h_0}\phi^h\left(x,\frac{h_0}{h}s\right)\Pi\right)(\mathrm{Id}+s\Pi)^{-1} & 0\\ \left[\frac{h}{h_0}\partial_{\tau_1}\phi^h\left(x,\frac{h_0}{h}s\right)\quad\frac{h}{h_0}\partial_{\tau_2}\phi^h\left(x,\frac{h_0}{h}s\right)\right](\mathrm{Id}+s\Pi)^{-1} & \partial_t\phi^h\left(x,\frac{h_0}{h}s\right)\end{bmatrix}.$$

Changing the variable $s=t\frac{h}{h_0}$, we obtain:

$$\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)=\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h(x,t)\vec{n}(x)\right)M_2, \tag{5.4}$$

where:

$$M_2=\begin{bmatrix}\left(\mathrm{Id}+\frac{h}{h_0}\phi^h(x,t)\Pi\right)(\mathrm{Id}+t\frac{h}{h_0}\Pi)^{-1} & 0\\ \left[\frac{h}{h_0}\partial_{\tau_1}\phi^h(x,t)\quad\frac{h}{h_0}\partial_{\tau_2}\phi^h(x,t)\right](\mathrm{Id}+t\frac{h}{h_0}\Pi)^{-1} & \partial_t\phi^h(x,t)\end{bmatrix}.$$

Therefore:

$$\det\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right) = \det\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h(x,t)\vec{n}(x)\right)$$
$$\cdot\det\left[\left(\mathrm{Id}+\frac{h}{h_0}\phi^h(x,t)\Pi\right)\left(\mathrm{Id}+t\frac{h}{h_0}\Pi\right)^{-1}\right]\partial_t\phi^h(x,t),$$

and according to the incompressible condition for the situation with growth, there follows:

$$\begin{aligned}1 &= \det\left(\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)(\sqrt{g})^{-1}\right)\\ &= \partial_t\phi^h(x,t)\det\left(\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h(x,t)\vec{n}\right)(\sqrt{g})^{-1}\right)\\ &\qquad\cdot\det\left[\left(\mathrm{Id}+\frac{h}{h_0}\phi^h(x,t)\Pi\right)\left(\mathrm{Id}+t\frac{h}{h_0}\Pi\right)^{-1}\right].\end{aligned}$$

Hence the modification ϕ^h obeys the following ODE:

$$\begin{cases}\partial_t\phi^h(x,t) = f(x,\phi^h(x,t),t),\\ \phi^h(x,0)=0.\end{cases}\tag{5.5}$$

Here:

$$\begin{aligned}f(x,y,t)^{-1} &= \det\left(\nabla u_c^h\left(x+\frac{h}{h_0}y\vec{n}\right)(\sqrt{g})^{-1}\right)\\ &\qquad\cdot\det\left[\left(\mathrm{Id}+\frac{h}{h_0}y\Pi\right)\left(\mathrm{Id}+t\frac{h}{h_0}\Pi\right)^{-1}\right].\end{aligned}\tag{5.6}$$

By theory of families of solutions of parameter dependent ODEs, for each $x\in S$, there exists a unique solution $\phi^h(x,t)\in\mathcal{C}(-h_0/2,h_0/2)$ for each $x\in S$, $t\in(-h_0/2,h_0/2)$, and it also satisfies the following bounds:

$$\left|\phi^h(x,t)-t\right| = o(h),\quad \left|\partial_t\phi^h-1\right| = o(h),\quad \left|\partial_\tau\phi^h(x,t)\right| = o(h),\quad \forall\tau\in T_xS.\tag{5.7}$$

Please refer to Appendix for detailed argument. As we have already seen in the proof of Theorem 2.1, the crucial step to investigate the rescaled energy functional is to find $\nabla u^h(x+th/h_0\vec{n})$. Observe that:

$$\begin{aligned}&\nabla u^h\left(x+t\frac{h}{h_0}\vec{n}\right)-\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right) = \nabla u_c^h\left(x+\frac{h}{h_0}\phi^h(x,t)\vec{n}\right)M_2-\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)\\ &= \left(\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h(x,t)\vec{n}\right)-\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)\right)M_2+\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)(M_2-\mathrm{Id}).\end{aligned}$$

On one hand, in view of (5.7), there follows:

$$\begin{aligned}&\nabla u_c^h\left(x+\frac{h}{h_0}\phi^h(x,t)\vec{n}\right)-\nabla u_c^h\left(x+t\frac{h}{h_0}\vec{n}\right)\\ &= \left(\phi^h-t\right)\frac{h}{h_0}\left[\partial_{\tau_1}\vec{N}-\nabla y\partial_{\tau_1}\vec{n}\quad \partial_{\tau_2}\vec{N}-\nabla y\partial_{\tau_2}\vec{n}\quad d\right]+o(h)=o(h).\end{aligned}$$

On the other hand, notice that:

$$M_2-\mathrm{Id} = \begin{bmatrix}\left(\mathrm{Id}+\frac{h}{h_0}\phi^h(x,t)\Pi\right)\left(\mathrm{Id}+t\frac{h}{h_0}\Pi\right)^{-1}-\mathrm{Id} & 0\\ \frac{h}{h_0}\left[\partial_{\tau_1}\phi^h(x,t)\quad \partial_{\tau_2}\phi^h(x,t)\right](\mathrm{Id}+t\Pi)^{-1} & \partial_t\phi^h(x,t)-1\end{bmatrix}.$$

Also, in view of (5.7) we obtain $|M_2 - Id| = o(h)$. Hence:

$$\begin{aligned}\nabla u^h\left(x + t\frac{h}{h_0}\vec{n}\right) &= \nabla u^h_c\left(x + t\frac{h}{h_0}\vec{n}\right) + o(h)\\ &= Q + t\frac{h}{h_0}\left[\partial_{\tau_1}\vec{N} - \nabla y \partial_{\tau_1}\vec{n} \quad \partial_{\tau_2}\vec{N} - \nabla y \partial_{\tau_2}\vec{n} \quad d\right] + o(h).\end{aligned}$$

As in the proof of Theorem 2.1 part (b), we may obtain the the upper bound through formal Taylor expansion. □

Lemma 5.2. *For each $\mathcal{C}^3$ isometric immersion y of $[g_{\alpha\beta}]$, there exists a sequence of recovery deformation $u^h \in \mathcal{C}^1(S^h;\mathbb{R}^3)$ such that part (b) in Theorem 2.2 holds.*

Proof. By the definition of $\mathcal{Q}_2^{In}$ and positive definiteness of $\mathcal{Q}_3$, there exists L^2 vector field $d : S \longrightarrow \mathbb{R}^3$ such that:

$$\begin{aligned}&\mathcal{Q}_2^{In}\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right)\\ &\qquad = \mathcal{Q}_3\left(\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\Pi\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1} + Q^T d\otimes\vec{n}\right),\end{aligned}$$

with $\mathrm{Tr}\left((\sqrt{[g_{\alpha\beta}]})^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1} + Q^T d\otimes\vec{n}\right) = 0$. From trace zero condition, regularity of S and y, we obtain $(Q^T d)\cdot\vec{n} \in \mathcal{C}^1(S)$. Let $\{d^n\}$ be a sequence of functions in $\mathcal{C}^1(S)$ which converge to d in $L^2(S,\mathbb{R}^3)$ with $(Q^T d^n)\cdot\vec{n} = (Q^T d)\cdot\vec{n}$. Notice that d^n satisfies the trace zero condition as well and:

$$\begin{aligned}&\lim_{n\to\infty}\int_S \mathcal{Q}_3\left(\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1} + (Q^T d^n)\otimes\vec{n}\right)\\ &\qquad = \int_S \mathcal{Q}_3\left(\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1} + (Q^T d)\otimes\vec{n}\right).\end{aligned}$$

Applying Lemma 5.1 to y, d^n and taking a diagonal sequence yields the required recovery sequence u^h. □

Appendix A. Existence of ϕ^h and the bounds (5.7)

Lemma A.1. *If h is sufficiently small, then for each $x \in S$ the ODE (5.5) has a unique solution $\phi^h(x,\cdot) : (-h_0/2, h_0/2) \longrightarrow \mathbb{R}$ and:*

$$\left|\phi^h(x,t)\right| < h_0 \qquad \forall x \in S, \quad \forall t \in \left(-\frac{h_0}{2}, \frac{h_0}{2}\right). \tag{A.1}$$

Moreover, ϕ^h is $\mathcal{C}^1$ regular in x and its tangential gradient $\nabla_{tan}\phi^h(x,t)$ is bounded:

$$\left|\partial_\tau\phi^h(x,t)\right| < C|\tau|, \qquad \forall t \in \left(-\frac{h_0}{2}, \frac{h_0}{2}\right) \quad \forall x \in S \quad \forall \tau \in T_xS. \tag{A.2}$$

Proof. **1.** Recall $f : S \times (-h_0, h_0) \times (-h_0/2, h_0/2) \longrightarrow \mathbb{R}$, defined in (5.6), has the following expression:

$$f(x,y,t) = \left(\det\left(\nabla u^h_c\left(x + y\frac{h}{h_0}\vec{n}\right)(\sqrt{g})^{-1}\right)\det\left[\left(\mathrm{Id} + y\frac{h}{h_0}\Pi\right)\left(\mathrm{Id} + t\frac{h}{h_0}\Pi\right)^{-1}\right]\right)^{-1}.$$

First, we notice that $f, \nabla_x f$ and $\nabla_y f$ are Lipschitz in y. Note that:

$$\det\left[\left(\mathrm{Id}+y\frac{h}{h_0}\Pi\right)\left(\mathrm{Id}+t\frac{h}{h_0}\Pi\right)^{-1}\right]=1+\frac{h}{h_0}(y-t)\mathrm{Tr}\Pi+o(h).$$

Hence:

$$f(x,y,t)^{-1}=(1+o(h))\left(1+\frac{h}{h_0}(y-t)\mathrm{Tr}(\Pi)+o(h)\right)=1+\frac{h}{h_0}(y-t)\mathrm{Tr}(\Pi)+o(h),$$

which implies:

$$f(x,y,t)=\left(1+\frac{h}{h_0}(y-t)\mathrm{Tr}(\Pi)+o(h)\right)^{-1}. \tag{A.3}$$

Thus, for sufficiently small h, there follows that:

$$\frac{1}{2}<f(x,y,t)<2, \qquad \forall x\in S \quad \forall y\in(-h_0,h_0) \quad \forall t\in\left(-\frac{h_0}{2},\frac{h_0}{2}\right). \tag{A.4}$$

2. We now study the derivative $\nabla_x f(x,y,t)$ and $\nabla_y f(x,y,t)$ of f. On one hand, for each $\tau\in T_xS$:

$$\nabla_x f(x,y,t)\tau=\frac{-h/h_0(y-t)\partial_\tau\left(\mathrm{Tr}(\Pi)\right)+o(h)}{\left(1+h/h_0(y-t)\mathrm{Tr}(\Pi)+o(h)\right)^2}.$$

On the other hand:

$$\nabla_y f(x,y,t)=\frac{-h/h_0\mathrm{Tr}(\Pi)+o(h)}{\left(1+h/h_0(y-t)\mathrm{Tr}(\Pi)+o(h)\right)^2}.$$

Thus, we may choose h so small that:

$$\|\nabla_x f(x,y,t)\|_\infty+\|\nabla_y f(x,y,t)\|_\infty\leq 1. \tag{A.5}$$

Consider the Banach space $B=\mathcal{C}(S\times(-\frac{h_0}{2},\frac{h_0}{2});(-h_0,h_0))$ equipped with the L^∞ norm and define the operator $T:B\longrightarrow B$:

$$(Tu)(x,t)=\int_0^t f(x,u(x,s),s)\mathrm{d}s, \quad \forall u\in B.$$

By (A.4), $-h_0<(Tu)(x,t)<h_0$ and so indeed $Tu\in B$. By (A.5), for each $u,v\in B$:

$$\begin{aligned}|(Tu-Tv)(x,t)|&=\left|\int_0^t f(x,u(x,s),s)\mathrm{d}s-\int_0^t f(x,v(x,s),s)\mathrm{d}s\right|\\&\leq\int_0^{|t|}|f(x,u(x,s),s)-f(x,v(x,s),s)|\,\mathrm{d}s\\&\leq\frac{1}{2}h_0\|u-v\|_\infty\leq\frac{1}{2}\|u-v\|_\infty.\end{aligned}$$

In view of the Banach fixed point theorem, there exists a unique $\phi^h\in B$, such that $T\phi^h=\phi^h$, which is equivalent to ϕ^h solving equation (5.5). Also (A.1) is a direct consequence of the definition of the space B.

3. Our next goal is to show that ϕ^h is $\mathcal{C}^1$ regular in x. For a fixed $\tau\in T_xS$, define:

$$S(u,v)(x,t)=\int_0^t\nabla_x f(x,u(x,s),s)\tau+\nabla_y f(x,u(x,s),s)v(x,s)\mathrm{d}s.$$

Let $u_0(x,t)=0$ and $v_0(x,t)=\partial_\tau u_0(x,t)=0$, and define sequences $(u_k),(v_k)$ iteratively as follows:

$$u_{k+1}=Tu_k,\ v_{k+1}=S(u_k,v_k),$$

so that $\lim_{k\to\infty} u_k = \phi^h$. We will now show that:

$$\lim_{k\to\infty} v_k = \partial_\tau \phi^h. \tag{A.6}$$

Note first that:

$$\partial_\tau (Tu)(x,t) = \int_0^t \nabla_x f(x,u(x,s),s)\tau + \nabla_y f(x,u(x,s),s)\partial_\tau u(x,s)\mathrm{d}s = S(u,\partial_\tau u). \tag{A.7}$$

In fact, an inductive argument shows that for each $k \in \mathbb{N}$, $v_k(x,t) = \partial_\tau u_k(x,t)$, because:

(i). For $k = 0$, this identity holds by the definition of v_0.
(ii). If $v_k = \partial_\tau u_k$ then, by (A.7): $v_{k+1} = S(u_k, v_k) = S(u_k, \partial_\tau u_k) = \partial_\tau (Tu_k) = \partial_\tau u_{k+1}$.

Now, for each $v \in \mathcal{C}(S \times (-h_0/2, h_0/2))$, define:

$$(Rv)(x,t) = S(\phi^h, v)(x,t) = \int_0^t \nabla_x f(x,\phi^h(x,s),s)\tau + \nabla_y f(x,\phi^h(x,s),s)v(x,s)\ \mathrm{d}s.$$

For any two v_1, v_2, we have:

$$|(Rv_1 - Rv_2)(x,t)| = \left|\int_0^t \nabla_y f(x,\phi^h(x,s),s)(v_1(x,t) - v_2(x,t))\right| \le \frac{1}{2}\|v_1 - v_2\|_\infty.$$

Thus R is a contraction, and hence it has a fixed point z. Furthermore notice that:

$$\begin{aligned} v_{k+1} - Rv_k =& S(u_k, v_k) - S(\phi^h, v_k) \\ =& \int_0^t \nabla_x f(x,u_k(x,s),s)\tau + \nabla_y f(x,u_k(x,s),s)v_k(x,s)\mathrm{d}s \\ & \qquad - \int_0^t \nabla_x f(x,\phi^h(x,s),s)\tau + \nabla_y f(x,\phi^h(x,s),s)v_k(x,s)\mathrm{d}x \\ \le& L_1\|u_k - \phi^h\|_\infty + L_2\|u_k - \phi^h\|_\infty \|v_k\|_\infty. \end{aligned}$$

where L_1 and L_2 are the Lipschitz constants of $\nabla_x f(x,y,t)$ and $\nabla_y f(x,y,t)$ with respect to y. Since $\|u_k - \phi^h\|_\infty \to 0$ as $k \to \infty$, we may use Ostrowski's theorem on approximate iteration [18], and conclude that $v_k \to z$ uniformly in $S \times (-h_0/2, h_0/2)$. Therefore, there must be $z = \partial_\tau \phi^h(x,t)$ and (A.6) follows.

4. We devote the following to proving (A.2). This will be a consequence of (A.6) and the boundedness of the sequence v_k. Indeed, by (A.5):

$$\begin{aligned} |v_1(x,t)| &= \left|\int_0^t \nabla_x f(x,u_0(x,s),s)\tau + \nabla_y f(x,u_0(x,s),s)v_0(x,s)\mathrm{d}s\right| \le \frac{1}{2}h_0, \\ |v_2(x,t)| &= \left|\int_0^t \nabla_x f(x,u(x,s),s)\tau + \nabla_y f(x,u(x,s),s)v_1(x,s)\mathrm{d}s\right| \\ &\le \int_0^{|t|} 1 + \frac{1}{2}h_0 \mathrm{d}s \le \frac{1}{2}h_0 + \left(\frac{1}{2}h_0\right)^2. \end{aligned}$$

Inductively, we obtain $\|v_k\|_\infty \le \sum_{i=1}^k (h_0/2)^i$. Thus:

$$\|v_k\|_\infty \le \frac{1}{2 - h_0},$$

and so (A.2) follows. □

Lemma A.2. *In the framework of Lemma A.1, the equalities in (5.7) hold.*

Proof. By (A.3), we see that:

$$\partial_t \phi^h(x,t) - 1 = -\frac{h}{h_0}\left(\phi^h - t\right)\mathrm{Tr}(\Pi) + o(h), \tag{A.8}$$

which implies:

$$|\partial_t \phi^h(x,t) - 1| = o(1), \tag{A.9}$$

and:

$$\left|\phi^h - t\right| = \left|\int_0^t \left(\partial_t \phi^h(x,s) - 1\right) \mathrm{d}s\right| = o(1). \tag{A.10}$$

Using the estimate (A.10) in (A.8), we obtain the second equality in (5.7). We may substitute the second equality to (A.10) and get the first equality in (5.7). Concerning the tangential derivative $\partial_\tau \phi^h$, observe that:

$$\partial_\tau \phi^h(x,t) = \partial_\tau\left(\int_0^t f(x,\phi^h(x,s),s)\mathrm{d}s\right) = \int_0^t \partial_\tau\left[\left(1+\frac{h}{h_0}(\phi^h - s)\mathrm{Tr}(\Pi) + o(h)\right)^{-1}\right]\mathrm{d}s$$

$$= \int_0^t \frac{-h/h_0 \partial_\tau \phi^h \mathrm{Tr}(\Pi) - h/h_0(\phi^h - s)\nabla\left(\mathrm{Tr}(\Pi)\right) + o(h)}{\left(1 + h/h_0(\phi^h - s)\mathrm{Tr}(\Pi) + o(h)\right)^2}\mathrm{d}s.$$

Using (A.2), the first equality in (5.7) and a bootstrap argument, we see that indeed:

$$|\partial_\tau \phi^h(x,t)| = o(h).$$

□

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University of Minnesota, Department of Mathematics, 206 Church St. S.E., Minneapolis, MN 55455
E-mail address: huilimath@umn.edu