RESEARCH STATEMENT

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My research consists of two interrelated parts. The first part lies in the field of the mathematical theory of nonlinear elasticity, and it concerns the rigorous derivation of theories for elastic shells. The second part concerns modeling and analyzing of shells with residual stresses. The approach for both parts is based on the refined methods in Calculus of Variations (notably the so-called Γ-convergence) and a combination of the arguments in modern Mathematical Analysis and Riemannian Geometry.

1. The Rigorous Derivation of Theories for Elastic Shells

1.1. **Background.** In the context of *Mathematical Theory of Elasticity*, the derivation of thin shell models is a fundamental question with a long history. Despite a large body of engineering literature, little is known about the mathematically rigorous justification of various plate and shell theories. Of even more concern is that some of the existing theories seem to be incompatible with each other. Recently, substantial analytical progress has been made possible due to the seminal work of Friesecke, James and Müller [11, 12], followed by other observations and results.

Given a 2-dimensional surface S in $\mathbb{R}^3$, define a family of thin shells:

$$S^h = \left\{ z = x + t\vec{n}(x); x \in S, -g_1^h(x) < t < g_2^h(x) \right\}, \tag{1}$$

where $\vec{n}(x)$ is the unit normal to S at the point x, and g_1^h, g_2^h are two sequences of positive functions representing the shell's boundary.

The total energy of a deformation $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ is given by:

$$J^h(u^h) = \frac{1}{h}\int_{S^h} W\left(\nabla u^h\right) - \frac{1}{h}\int_{S^h} f^h \cdot u^h, \tag{2}$$

where $W : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}_+$ is the elastic energy density function, satisfying physically relevant conditions, and $f^h \in L^2(S^h, \mathbb{R}^3)$ represents an external force acting on the shell.

In view of (2), now one wants to study the asymptotic behavior of J^h as $h \to 0$. Since the conditions for W generally imply that the first term in $J^h(u^h)$ is non-convex in its argument u^h while the second term in $J^h(u^h)$ is linear, the main variational analysis concerns the elastic energy:

$$I^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h).$$

and it attempts computing the Γ-limit I_β of the sequence of scaled energies $h^{-\beta}I^h$, where the exponent β is determined by the scaling α of the external force f^h. Namely, one can prove that if $f^h \sim h^\alpha$, the elastic energy $I^h(u^h)$ at minimizers of J^h scales like h^β, where $\beta = \alpha$ if $0 \le \alpha \le 2$, and $\beta = 2\alpha - 2$ if $\alpha \ge 2$. The limiting energy I_β to be determined plays hence the role of the 2d counterpart of the 3d energy functionals I^h, since Γ-convergence [7, 8] guarantees that the global minimizers of the 3d energy functionals converge to the global minimizers of the 2d limiting energy.

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When S^h is a plate with uniform thickness ($S \subset \mathbb{R}^2$ and $g_1^h = g_2^h = h/2$), such Γ-convergence was first established by LeDret and Raoult [14] for $\beta = 0$, then by Friesecke, James and Müller [11, 12] for all $\beta \geq 2$. In the case of $0 < \beta < 5/3$, the related result was obtained by Conti and Maggi [6]. The regime $5/3 \leq \beta < 2$ remains open and is proposed to be relevant to crumpling of elastic sheets.

If S^h is a shell with uniform thickness (S is an arbitrary surface and $g_1^h = g_2^h = h/2$), the Γ-convergence was first obtained in [15] for $\beta = 0$. The model is that of a *membrane shell* and the limit I_0 depends only on the stretching and shearing produced by the deformation on the surface S. Another study is due to Friesecke, James, Mora and Müller in [10], who analyzed the case $\beta = 2$. This scaling corresponds to a *flexural shell model*, where the only admissible deformations are those preserving the metric on S. The energy I_2 depends then on the change of curvature produced by the deformation. Further, Lewicka, Mora and Pakzad studied the situation $\beta \geq 4$ in [18]. For $\beta = 4$, the Γ-limit obtained therein is a generalization of the *von Kármán theory* for plates, which for $\beta > 4$ reduces to the linearized flexural shell model.

1.2. **The von Kármán theory for incompressible elastic shells.** In this project, we take into account the additional constraint of incompressibility, which is frequently encountered as a property of elastic materials. Consider uniform thickness shells ($g_1^h = g_2^h = h/2$) with the associated elastic energy given by:

$$I^h(u^h) = \frac{1}{h}\int_{S^h} W_{In}(\nabla u^h), \tag{3}$$

where the incompressible energy density $W_{In} : \mathbb{R}^{3\times 3} \longrightarrow [0, \infty]$ has the form:

$$W_{In}(F) = \begin{cases} W_c(F), & \text{if } \det F = 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

where W_c satisfies the physical relevant conditions as W dose as we introduced in the background.

We proved in [24] that when $\beta = 4$, whenever the midsurface S is assumed to enjoy the following approximation property: $\mathcal{C}^3$ first order infinitesimal isometries are dense in the space of $W^{2,2}$ infinitesimal isometries, the scaled energy functional $h^{-4}I^h(u^h)$ Γ-converges to:

$$\mathcal{I}(V, B_{tan}) = \frac{1}{2}\int_S \mathcal{Q}_2^{In}\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan}\right) + \frac{1}{24}\int_S \mathcal{Q}_2^{In}\left(x, (\nabla(A\vec{n}) - A\nabla\vec{n})_{tan}\right), \tag{4}$$

where the quadratic form $\mathcal{Q}_2^{In}$ is defined as:

$$\mathcal{Q}_2^{In}(x, F_{tan}) = \min_{d\in\mathbb{R}^3}\{\mathcal{Q}_3(F_{tan} + d\otimes\vec{n} + \vec{n}\otimes d); \mathrm{Tr}(F_{tan} + d\otimes\vec{n} + \vec{n}\otimes d) = 0\}, \tag{5}$$

and $\mathcal{Q}_3 = \nabla^2 W(\mathrm{Id})(F, F)$. The class of surfaces with the desired property includes: subsets of $\mathbb{R}^2$, strictly convex $\mathcal{C}^{5,\alpha}$ surfaces, developable $\mathcal{C}^{4,1}$ surfaces and rotationally invariant $\mathcal{C}^3$ surfaces.

Moreover, when $S = \Omega \subset \mathbb{R}^2$, the energy functional (4) becomes:

$$\mathcal{I}(w, v) = \frac{1}{2}\int_\Omega \mathcal{Q}_2^{In}\left(\mathrm{sym}\nabla w + \frac{1}{2}\nabla v\otimes\nabla v\right) + \frac{1}{24}\int_\Omega \mathcal{Q}_2^{In}\left(\nabla^2 v\right),$$

and its corresponding Euler-Lagrange equations for isotropic materials are given by:

$$\begin{cases} \frac{\mu}{3}\Delta^2 v = [v, \Phi], \\ \Delta^2\Phi = -\frac{3\mu}{2}[v, v], \end{cases} \tag{6}$$

which are the incompressible version of the classical von Kármán equations, obtained formally in the limit of Poisson's ratio $\nu \to 1/2$. Thus, this project rigorously justified the classical version of von Kármán equations. Our analysis relies on the methods and extends the results of [5, 12, 18].

1.3. **The von Kármán theory for shells with variable thickness.** Here, we study the case $\beta = 4$ with given sequences g_1^h, g_2^h of positive $\mathcal{C}^1$ functions with the property that:

$$\lim_{h\to 0} g_1^h/h = g_1 \quad \text{and} \quad \lim_{h\to 0} g_2^h/h = g_2 \quad \text{in } \mathcal{C}^1(S),$$

for some positive $\mathcal{C}^1$ functions $g_1, g_2 : S \to \mathbb{R}_+$. In [25], we proved that the scaled elastic energy $h^{-4}I^h(u^h)$ Γ-converges to the 2d energy:

$$\begin{aligned}\mathcal{I}(V, B_{tan}) = \frac{1}{2}\int_S (g_1+g_2)\mathcal{Q}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan} - \frac{1}{2}\text{sym}\left(A\nabla((g_2-g_1)\vec{n})\right)\right) \\ + \frac{1}{24}\int_S (g_1+g_2)^3\mathcal{Q}_2\left(x, (\nabla(A\vec{n}) - A\nabla\vec{n})_{tan}\right),\end{aligned} \tag{7}$$

for $V \in \mathcal{V}$ and $B_{tan} \in \mathcal{B}$. The space $\mathcal{V}$ is the space of $W^{2,2}$ first order infinitesimal isometries, which consists of vector fields $V \in W^{2,2}(S, \mathbb{R}^3)$ satisfying the property that there exists a matrix field $A \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ with:

$$\partial_\tau V(x) = A(x)\tau \quad \text{and} \quad A(x)^T = -A(x), \quad \forall a.e.\ x \in S,\ \forall \tau \in T_xS.$$

The finite strain space $\mathcal{B}$ consists of all symmetric matrix fields which are L^2 limits of symmetric gradients of $W^{1,2}$ vector fields on S. In (7), the quadratic forms $\mathcal{Q}_2(x,\cdot)$ are defined as:

$$\mathcal{Q}_2(x, F_{tan}) = \min\left\{\mathcal{Q}_3(\tilde{F}); \left(\tilde{F} - F\right)_{tan} = 0\right\}, \quad \mathcal{Q}_3(F) = \nabla^2 W(\text{Id})(F,F). \tag{8}$$

As introduced in [19], (7) is called the *generalized von Kármán functional.* This research meets the need of deriving appropriate limiting theories with more general thickness given by g_1^h, g_2^h. It also extends the the result given in [19], where the thickness is given by $g_1^h = hg_1$ and $g_2^h = hg_2$ for some positive Lipschitz functions g_1 and g_2.

1.4. **Further research projects.** All the completed projects consider the convergence of minimizing equilibria. While there are some results about the convergence of non-minimizing equilibria under the von Kármán setting by Müller, Pakzad, Lewicka, Mora and Scardia in [16, 23, 22], there is still the need of the parallel research under the Kirchhoff theory setting, which will be one of my further research topics.

I will furthermore investigate the dynamic equations and the 2d theories they imply. One recent work in this direction is due to Abels, Mora and Müller in [1].

2. Modeling of Shells with Residual Stress

2.1. **Background.** Wrinkles and ripples of thin films frequently occur in biological, physical and chemical situations (e.g. at edges of flowers, leaves and torn plastic sheets), which assume non-trivial shapes even in the absence of external forces. The mechanism of such phenomena has been studied in [9, 13], where it is proposed that due to growth or other causes, a local and heterogeneous incompatibility of strains is produced, resulting in local elastic stresses. To fully relieve the stresses, the thin structure S^h (S^h is defined as in section 1 with the thickness $g_1^h = g_2^h = h/2$) strives to realize the metric g^h. However, due to the geometric restriction of space, in general g^h cannot be realized and hence the shell S^h will be settled with shapes closest to the realization of such metric. Generally, we will consider the asymptotic behavior of the energy functional:

$$I^h(u^h) = \frac{1}{h}\int_{S^h} W\left(\nabla u^h \left(\sqrt{g^h}\right)^{-1}\right), \tag{9}$$

defined on the set of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ of S^h. Here W enjoys the physical relevant conditions as well and these conditions make $I^h(u^h)$ is comparable to:

$$I^h_W(u^h) = \frac{1}{h}\int_{S^h} \mathrm{dist}^2\left(\nabla u^h \left(\sqrt{g^h}\right)^{-1}, SO(3)\right), \tag{10}$$

and hence it measures how well the metric g^h is realized by the deformation u^h.

Recently, researchers have adopted the approach of Γ-convergence described in section 1 and obtained several results. In [20], Lewicka and Pakzad studied pre-strained plates, i.e. $S \subset \mathbb{R}^2$ with metric g^h independent of thickness h in the energy scaling regime given by $I^h(u^h) \leq Ch^2$ and obtained a variant of Kirchhoff plate theory. In [17], Lewicka, Mahadevan and Pakzad rigourously justified the von Kármán-like equations for plates, which were first obtained via asymptotic expansions by Liang and Mahadevan [21].

2.2. **The Kirchhoff theory of elastic pre-strained shells.** We first extends the discussion in [20] to shells and then includes a parallel discussion for the incompressible case. To be more specific, our first result is that for general surface S, with the same scaling of energy functionals and the same condition of metric g^h as in [20], the rescaled energy functional $h^{-2}I^h(u^h)$ Γ-converges, as the thickness $h \to 0$, to the functional:

$$\mathcal{I}(y) = \frac{1}{24}\int_S \mathcal{Q}_2\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right)\mathrm{d}x.$$

Here, the term $(\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\nabla\vec{n}$ measures the relative bending with respect to the desired metric. Furthermore, taking into account the incompressibility of reference shell with respect to metric g^h, i.e. $\det\left(\nabla u^h\left(\sqrt{g^h}\right)^{-1}\right) = 1$, we prove that the Γ-limit of rescaled energy functional $h^{-2}I^h(u^h)$ has the form:

$$\mathcal{I}_{In}(y) = \frac{1}{24}\int_S \mathcal{Q}_2^{In}\left(x, \left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - [g_{\alpha\beta}]\,\nabla\vec{n}\right)\left(\sqrt{[g_{\alpha\beta}]}\right)^{-1}\right),$$

where $\mathcal{Q}_2^{In}(x,\cdot)$ is defined as in (5) while $\mathcal{Q}_2(x,\cdot)$ as in (8).

2.3. **Derivation of (9) from the discrete (atomistic) Model.** This is an ongoing project in collaboration with Lewicka. We investigate the mathematical validity of the model (9). Though there are some experimental support, it is still necessary to derive it from the atomistic point of view. The starting point is the total energy functional F_ε on the ε-grid defined as:

$$F_\varepsilon(u) = \sum_{\alpha,\beta\in\varepsilon\mathbb{Z}^3, [\alpha,\beta]\subset S^h} f_\varepsilon(\alpha,\beta,u(\alpha)-u(\beta)), \tag{11}$$

where $g_\varepsilon(\alpha,\beta,u(\alpha)-u(\beta))$ weighs the pairwise interaction between points defined as:

$$f_\varepsilon(\alpha,\beta,d) = \varepsilon\psi\left(\frac{|\alpha-\beta|}{\varepsilon}\right)\left(|d| - \langle(\beta-\alpha), g^h(\alpha)(\beta-\alpha)\rangle^{1/2}\right)^2,$$

here ψ is a cut-off function illustrating the interaction radius. Based on results in [2], we are working on the derivation of (9) from (11) via Γ-convergence.

2.4. **Further research projects.** Besides the ongoing research with Lewicka, I am also interested in a project based on the analysis in [17]. I have already established the convergence of equilibria for compressible shells with uniform thickness and I plan to study the model under the varying thickness and incompressible setting. All the obtained results will constitute [27].

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