The von Kármán theory for incompressible elastic shells

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Elastic Shells

Elastic Shells

$S \subset \mathbb{R}^3$ is a 2d surface and $\vec{n}(x)$ is its unit normal vector at x:

$$S^h = \left\{ z = x + t\vec{n}(x); x \in S, -\frac{h}{2} < t < \frac{h}{2} \right\}$$

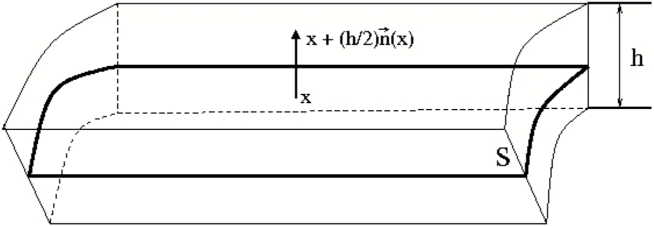

Figure: The shell S^h of midsurface S and small thickness h.

Incompressibility

- Characterization

$$S^h \text{ is incompressible, } u^h \text{ is its deformation}$$

$$\Downarrow$$

$$\det(\nabla u^h) = 1$$

- Some example: Rubber

Question

What is the corresponding von Kármán theory for incompressible shells?

Von Kármán Equations for Plates ($S \subset \mathbb{R}^2$)

A system of equations for out-of-plane displacement w and Airy stress potential Φ:

$$\begin{cases} D\Delta^2 w = h[\Phi, w] + f, \\ \Delta^2 \Phi = -\frac{1}{2}E[w, w], \end{cases}$$

where

Airy bracket: $[u, v] \equiv \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}$,

Young modulus: $E > 0$,

Poisson ratio: $\nu \in \left(-1, \frac{1}{2}\right)$,

Flexural rigidity: $D = \frac{Eh^3}{12(1-\nu^2)}$.

Justification and Generalization of von Kármán Theory

- Formal justification
  - Basic tool: Formal asymptotic expansions
    P.G. Ciarlet and P. Destuynder (1979).
  - The formal justification: P.G. Ciarlet (1980), Fox, Raoult and Simo (1993).
  - In the framework of growth: Ben Amar, Dervaux (2008).
- Rigorous justification
  - Basic tool: Γ-convergence
    First to justify the 2d theories from 3d: Le Dret and Raoult (1993).
  - An important rigidity result: Friesecke, James and Müller (2002).
  - The rigorous justification: Friesecke, James and Müller (2006).
  - Generalization to shells of arbitrary geometry: Lewicka, Mora and Pakzad (2010).

Scaled Elastic Energy

- Scaled elastic energy of deformation u^h of S^h

$$I^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h)$$

where $W : \mathbb{R}^{3\times 3} \longrightarrow [0, +\infty]$ obeys:

$$W(F) = \begin{cases} W_c(F), & \text{if } \det F = 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

$W_c : \mathbb{R}^{3\times 3} \to [0, +\infty]$ satisfies:

(i) $W_c(RF) = W_c(F)$, $\forall R \in SO(3)$, Frame invariance

(ii) $W_c(\mathrm{Id}) = 0$, Normalization

(iii) $W_c(F) \geq c\,\mathrm{dist}^2(F, SO(3))$ for $c > 0$, Growth

(iv) W_c is $\mathcal{C}^2$ in a neighborhood of $SO(3)$. Regularity

(W-H) — conditions (i)–(iv)

Γ-convergence

Definition of Γ-convergence

$\mathcal{F}^h, \mathcal{F} : X \longrightarrow \bar{\mathbb{R}}, \mathcal{F}^h \xrightarrow{\Gamma} \mathcal{F}$ iff:

(i) Lower bound: For each $x^h \to x$ in X: $\liminf_{h\to 0} \mathcal{F}^h(x^h) \geq \mathcal{F}(x)$.

(ii) Recovery sequence: For every $x \in X$, there exists $x^h \to x$, such that: $\lim_{h\to 0} \mathcal{F}^h(x^h) = \mathcal{F}(x)$.

Metatheorem

$\mathcal{F}^h \xrightarrow{\Gamma} \mathcal{F}$ + Compactness $\Longrightarrow$ The global minimizers of $\mathcal{F}^h$ converge to the global minimizers of $\mathcal{F}$.

Two Important Spaces

- Space of first order infinitesimal isometries $\mathcal{V}$:

$$\mathcal{V} = \{V \in W^{2,2}(S, \mathbb{R}^3), \mathrm{sym}\nabla V = 0\}.$$

  - Let $u_\varepsilon = id + \varepsilon V$, $\quad |\partial_\tau u_\varepsilon|^2 - |\tau|^2 = \mathcal{O}(\varepsilon^2) \quad \forall \tau \in T_x S.$

- Finite strain space $\mathcal{B}$:

$$\mathcal{B} = cl_{L^2}\{\mathrm{sym}\nabla w; w \in W^{1,2}(S, \mathbb{R}^3)\}.$$

The Generalized von Kármán Functional on S

$$I(V,B_{tan}) = \frac{1}{2}\int_S \mathcal{Q}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan}\right) + \frac{1}{24}\int_S \mathcal{Q}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\right),$$

where

$$\mathcal{Q}_2(x,G_{tan}) = \min_{d\in\mathbb{R}^3}\{\mathcal{Q}_3(G_{tan} + d\otimes\vec{n} + \vec{n}\otimes d); \mathrm{Tr}(G_{tan} + d\otimes\vec{n} + \vec{n}\otimes d) = 0\},$$

$$\mathcal{Q}_3(G) = \nabla^2 W_c(\mathrm{Id})(G,G),$$

A is a skew symmetric matrix field on S, s.t. $\partial_\tau V = A\tau$ for each $\tau \in T_x S$.

- Stretching: $B_{tan} - \frac{1}{2}(A^2)_{tan}$.
- Bending: $(\nabla(A\vec{n}) - A\Pi)_{tan}$.

Surface S

S is a smooth compact connected oriented $2d$ surface embedded in $\mathbb{R}^3$, enjoying the approximation property (H).

$$\left.\begin{array}{l}\text{The space } \mathcal{V}_0 = \mathcal{C}^3(\bar{S}, \mathbb{R}^3) \cap \mathcal{V} \text{ is dense in } \mathcal{V} \text{ with respect} \\ \text{to the } W^{2,2}(S, \mathbb{R}^3) \text{ topology.}\end{array}\right] \qquad \text{(H)}$$

Surfaces enjoying condition (H) include:

- flat surfaces $S \subset \mathbb{R}^2$. (Easily seen by Korn's inequality)
- smooth rotationally invariant surfaces. (Direct check)
- strictly convex $\mathcal{C}^{5,\alpha}$ surfaces. (Lewicka, Mora and Pakzad)
- developable $\mathcal{C}^{4,1}$ surfaces without flat parts. (Hornung, Lewicka and Pakzad)

Theorem (Compactness and Lower Bound)

Let a sequence of deformations $u^h \in W^{1,2}(S^h)$ satisfy:

$$I^h(u^h) \leq Ch^4.$$

Then $\exists Q^h \in SO(3)$ and $\exists c^h \in \mathbb{R}^3$ such that for

$$y^h(x + t\vec{n}(x)) = (Q^h)^T u^h(x + th/h_0\vec{n}) - c^h : S^{h_0} \longrightarrow \mathbb{R}^3$$

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(ii) $V^h = \frac{1}{h}\fint_{-h_0/2}^{h_0/2} \big(y^h(x + t\vec{n}) - x\big)\,\mathrm{d}t \to V \in \mathcal{V}$ in $W^{1,2}(S)$,

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(iii) $\frac{1}{h}\mathrm{sym}\nabla V^h \rightharpoonup B_{tan} \in \mathcal{B}$ in $L^2(S)$,

(iv) $\liminf_{h\to 0} \frac{1}{h^4} I^h(u^h) \geq \mathcal{I}(V, B_{tan})$.

Theorem (Recovery Sequence)

For each $V \in \mathcal{V}$, $B_{tan} \in \mathcal{B}$, there exists $u^h \in \mathcal{C}^1(S^h, \mathbb{R}^3)$ such that:

(i) $y^h(x + t\vec{n}(x)) = u^h(x + th/h_0\vec{n}) \to x$ in $W^{1,2}(S^{h_0})$.

(ii) The scaled average displacements $V^h \to V$ in $W^{1,2}(S)$.

(iii) $\frac{1}{h}\mathrm{sym}\nabla V^h \to B_{tan}$ in $L^2(S)$.

(iv) $\lim_{h\to 0} \frac{1}{h^4} I^h(u^h) = \mathcal{I}(V, B_{tan})$.

where the definition for each quantity is exactly the same as in previous theorem.

Incompressible Plates ($S \subset \mathbb{R}^2$)

Two spaces for plane:

$$\mathcal{B} = \left\{ \mathrm{sym}\nabla w; w \in W^{1,2}(S, \mathbb{R}^2) \right\},$$

$$\mathcal{V} = \left\{ V = (ax_2 + b_1, -ax_1 + b_2, v_3)^T; a, b_1, b_2 \in \mathbb{R}, v_3 \in W^{2,2}(S, \mathbb{R}) \right\}.$$

The von Kármán functional for incompressible plates:

$$I(w, v) = \frac{1}{2} \int_S \mathcal{Q}_2 \left(\mathrm{sym}\nabla w + \frac{1}{2} \nabla v \otimes \nabla v \right) + \frac{1}{24} \int_S \mathcal{Q}_2 \left(\nabla^2 v \right).$$

where v is the scaled out-plane displacement, w is the scaled in-plane displacement.

Incompressible Plates ($S \subset \mathbb{R}^2$)

Euler-Lagrange equations of I for isotropic plates

$$\begin{cases} \Delta^2 v = 6[v, \Phi], \\ \Delta^2 \Phi = -\frac{3}{4}[v, v]. \end{cases}$$

The Airy stress potential $\Phi \in W^{2,2}(\Omega, \mathbb{R})$ satisfies:

$$\mathrm{cof}\nabla^2\Phi = \mathrm{sym}\nabla w + \frac{1}{2}\nabla v \otimes \nabla v + \left(\mathrm{div} w + \frac{1}{2}|\nabla v|^2\right) \mathrm{Id}.$$

Isotropic material:

$$\mathcal{Q}_3(G) = 2\mu|\mathrm{sym}G|^2 + \lambda|\mathrm{Tr}G|^2,$$

thus,

$$\mathcal{Q}_2(G_{tan}) = 2\mu\left(|\mathrm{sym}G_{tan}|^2 + |\mathrm{Tr}G_{tan}|^2\right).$$

The Recovery Sequence

Combination of arguments in :
Lewicka-Mora-Pakzad, Conti-Dolzmann

Density Results:

- (H) $\Rightarrow \mathcal{V}_0 = \mathcal{C}^3(\bar{S}, \mathbb{R}^3) \cap \mathcal{V}$ is dense in $\mathcal{V}$.
- $\mathcal{B}_0 = \{\mathrm{sym}\nabla w; w \in \mathcal{C}^\infty(\bar{S}, \mathbb{R}^3)\}$ is dense in $\mathcal{B}$.

$\Rightarrow$ Diagonal argument.

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$\Rightarrow$ Diagonal argument.

Lemma

For $\forall V \in \mathcal{V}_0$, $\forall B_{tan} \in \mathcal{B}_0$, $\exists u^h \in \mathcal{C}^1(S^h, \mathbb{R}^3)$, such that (i), (ii), (iii) and (iv) of Theorem (Recovery Sequence) hold.

The Recovery Sequence

Key points of the proof of Lemma:

- Scaled recovery sequence for V and B_{tan}: $y_c^h(x+t\vec{n})$,
- Modified sequence $y^h(x+t\vec{n}) = y_c^h(x+\varphi^h(x,t)\vec{n})$, or $u^h(x+s\vec{n}) = u_c^h(x+h/h_0\varphi^h(x,h_0 s/h)\vec{n})$,
- $\det\nabla u^h = 1 \Rightarrow$ ODE satisfied by $\varphi^h(x,t)$,
- ODE + bounds for $\det\nabla u_c^h \Rightarrow$ essential bounds for φ^h:

$$\left|\partial_t\varphi^h - 1\right| \leq Ch^3, \quad \left|\varphi^h - t\right| \leq Ch^3, \quad \left|\partial_\tau\varphi^h\right| \leq Ch^3,$$

- Main obstacle: Establishing the bounds.
  Solution: When analyzing $\det\nabla u_c^h$, instead of considering it directly, we consider $\det[(\nabla u_c^h)^T(\nabla u_c^h)]$, followed by a Taylor expansion.

Further Consideration

- Hierarchy of 2d theories for incompressible elastic plates/shells.
- Varying thickness

$$\left(-\frac{h}{2},\frac{h}{2}\right) \longrightarrow (-g_1^h, g_2^h)$$

for some general function g_1^h, g_2^h.

Thank you!