The Von Kármán Theory for Elastic Shells With Variable Thickness

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Outline

The derivation of thin shell models is a fundamental question in mathematical elasticity.
(Love, Kirchhoff, von Kármán, Ciarlet, Friesecke, Müller, James, Conti, Dolzmann, Mora, Pakzad, Lewicka)

$S \subset \mathbb{R}^3$ a 2d surface, normal vector $\vec{n}(x)$.
Thin shells:

$$S^h = \{z = x + t\vec{n}(x); x \in S, -g_1^h(x) < t < g_2^h(x)\}$$

where $g_i^h : S \longrightarrow \mathbb{R}_+$, for $i = 1, 2$.

Q: The shape the elastic shell assumes in response to an external force?

- Elastic energy of deformation u^h of S^h

$$E^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h)$$

where $W : \mathbb{R}^{3\times 3} \longrightarrow [0,\infty]$, satisfies:

(i) $W(RF) = W(F) \quad \forall R \in SO(3)$ Frame invariance,
(ii) $W(R) = 0 \quad \forall R \in SO(3)$ Normalization,
(iii) $W(F) \geq c\,\mathrm{dist}^2(F, SO(3))$, $c > 0$ Nondegeneracy.

- Total energy functional

$$J^h(u^h) = E^h(u^h) - \frac{1}{h}\int_{S^h} f^h \cdot u^h$$

- Main Question: what is argmin J^h?
- Basic tool: Γ-convergence.

Definition of Γ-convergence

$\mathcal{F}^h, \mathcal{F} : X \longrightarrow \bar{\mathbb{R}}, \mathcal{F}^h \xrightarrow{\Gamma} \mathcal{F}$ iff:

(i) For each $x^h \to x$ in X: $\liminf_{h\to 0} \mathcal{F}^h(x^h) \geq \mathcal{F}(x)$.

(ii) For every $x \in X$, there exists $x^h \to x$, such that: $\lim_{h\to 0} \mathcal{F}^h(x^h) = \mathcal{F}(x)$.

Metatheorem

$\mathcal{F}^h \xrightarrow{\Gamma} \mathcal{F} + \textit{Compactness} \Longrightarrow$ The global minimizers of $\mathcal{F}^h$ converge to the global minimizers of $\mathcal{F}$

Remark: $\mathcal{F}$ is lower semi-continuous.

- Total energy functional

$$J^h(u^h) = E^h(u^h) - \frac{1}{h}\int_{S^h} f^h \cdot u^h$$

- Main Question: Identify Γ-limit J of J^h or Γ-limit J_β of $h^{-\beta}J^h$ (argmin $J^h \to$ argmin J)
  - $f^h \sim h^\alpha \Longrightarrow E^h(u^h) \sim h^\beta$ at the minimizer of J^h.
    - $\beta = \alpha$, if $0 \leq \alpha \leq 2$.
    - $\beta = 2\alpha - 2$, if $\alpha \geq 2$.

- Total energy functional

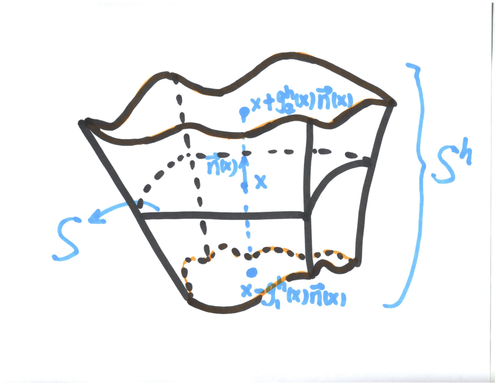
$$J^h(u^h) = E^h(u^h) - \frac{1}{h}\int_{S^h} f^h \cdot u^h$$

- Main Question: Identify Γ-limit I_β of $h^{-\beta}E^h$

Previous results ($\beta \geq 0$)

- Plate theories, $S \subset \mathbb{R}^2$, $g_1^h(x) = g_2^h(x) = h/2$
  - $\beta = 0$ LeDret and Raoult (1995)
  - $\beta \geq 2$ Friesecke, James and Müller (2002, 2006)
  - $0 < \beta < 5/3$ Conti and Maggi (2008)
  - $5/3 \leq \beta < 2$ Open, crumpling of elastic sheets
- Shell theories, general $S \subset \mathbb{R}^3$, $g_1^h(x) = g_2^h(x) = h/2$
  - $\beta = 0$ LeDret and Roult, Membrane model (1996)
  - $\beta = 2$ Friesecke, James, Mora and Müller, Flexural shell model (2003)
  - $\beta \geq 4$ Lewicka, Mora and Pakzad (2009) [LMP]
- von Kármán theory ($\beta = 4$) and $g_1^h = hg_1, g_2^h = hg_2$
  Lewicka, Mora and Pakzad (2009)

Shells with oscillating thickness



g_1^h, g_2^h are $\mathcal{C}^1$ s.t. $\exists$ $\mathcal{C}^1$ functions $g_1, g_2 : S \longrightarrow \mathbb{R}_+$ s.t.

$$\frac{g_i^h}{h} \to g_i \quad \text{in} \quad \mathcal{C}^1(S) \quad \textit{for} \quad i = 1, 2.$$

Theorem 1 (Compactness and lower bound)

Let a sequence of deformations $u^h \in W^{1,2}(S^h)$ satisfying:

$$E^h(u^h) \leq Ch^4.$$

Then $\exists Q^h \in SO(3)$ and $\exists c^h \in \mathbb{R}^3$ such that for $\tilde{u}^h(z) = (Q^h)^T u^h(z) + c^h$ the following hold:

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(i) $(\tilde{u}^h \circ s^h)(x + t\vec{n}) \to x$ in $W^{1,2}(S^*)$, as $h \to 0$,

(ii) $V^h \to V \in \mathcal{V}$ in $W^{1,2}(S)$, as $h \to 0$,

- $V^h = \frac{1}{h} \int_{-h_0/2}^{h_0/2} (\tilde{u}^h - id) \circ s^h(x + t\vec{n}(x))\mathrm{d}t,$
- $\mathcal{V}$ consists of all first order isometries, i.e.:

  for each $V \in \mathcal{V}$, $\mathrm{sym}\nabla V = 0$

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(iii) $\frac{1}{h}\mathrm{sym}\nabla V^h \rightharpoonup B_{tan} \in \mathcal{B}$ in $L^2(S)$,as $h \to 0$,

- the space of finite strains $\mathcal{B} = cl_{L^2}\{\mathrm{sym}\nabla w; w \in W^{1,2}(S, \mathbb{R}^3)\}$.

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(iv) $\liminf_{h\to 0} \frac{1}{h^4} E^h(u^h) \geq I(V, B_{tan})$

The generalized von Kármán functional on S

$$I(V,B_{tan}) = \frac{1}{2}\int_S (g_1+g_2)\mathcal{Q}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan} - \frac{1}{2}\mathrm{sym}\,\left(A\nabla((g_2-g_1)\vec{n})\right)\right)$$
$$+\frac{1}{24}\int_S (g_1+g_2)^3\;\mathcal{Q}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\right)$$

where

$$\mathcal{Q}_2(x,F_{tan}) = min\{\mathcal{Q}_3(\tilde{F}); (\tilde{F}-F)_{tan} = 0\}, \mathcal{Q}_3(F) = D^2W(\mathrm{Id})(F,F)$$

- Stretching: $B_{tan} - \frac{1}{2}(A^2)_{tan} - \frac{1}{2}\mathrm{sym}\,\left(A\nabla((g_2-g_1)\vec{n})\right)$
- Bending: $(\nabla(A\vec{n}) - A\Pi)_{tan}$

Crucial ingredient of the proof of Theorem 1

Rigidity Estimate (Friesecke, James and Müller 2002)

$\forall u \in W^{1,2}(\Omega, \mathbb{R}^3), \exists R \in SO(3)$ such that

$$\int_\Omega |\nabla u - R|^2 \leq C \int_\Omega \operatorname{dist}^2(\nabla u, SO(3))$$

and $C = C(\Omega)$.

Remark: $C(S^h)$ is of order h^{-2}.
Based on the Rigidity Estimate, we obtain compactness of the sequence u^h with controlled energy. The lower bound by $I(V, B_{tan})$ follows through formal Taylor expansion of $W(\nabla u)$ around Id.

Theorem 2 (Recovery Sequence)

For each $V \in \mathcal{V}$, $B_{tan} \in \mathcal{B}$, there exists $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ such that:

(i) $(u^h \circ s^h)(x + t\vec{n}) \to x$ in $W^{1,2}(S^*)$, as $h \to 0$,

(ii) $V^h \to V \in \mathcal{V}$ in $W^{1,2}(S)$, as $h \to 0$,

(iii) $\frac{1}{h}\mathrm{sym}\nabla V^h \to B_{tan} \in \mathcal{B}$ in $L^2(S)$, as $h \to 0$,

(iv) $\lim_{h\to 0} \frac{1}{h^4} E^h(u^h) = I(V, B_{tan})$.

where the definition for each quantity is exactly the same as in previous theorem, with $\tilde{u}^h = u^h$.

Construction of the Recovery Sequence

Construction of the Recovery Sequence

$$
\begin{aligned}
u^h \circ s^h(x + t\vec{n}) = x &+ \frac{1}{2}\left(g_2^h - g_1^h\right)\vec{n} + hv^h(x) + h^2 w^h(x) \\
&+ \frac{t}{h_0}\left(g_1^h + g_2^h\right)\vec{n}(x) \\
&+ \frac{t}{h_0} h\left(g_1^h + g_2^h\right)\left(\Pi v_{tan}^h - \nabla(v^h\vec{n})\right)(x) \\
&- \frac{t}{h_0} h^2\left(g_1^h + g_2^h\right)\left(\nabla w^h\right)^T \vec{n}(x) \\
&+ \frac{t}{h_0} h^2\left(g_1^h + g_2^h\right) d^{0,h}(x) \\
&+ \frac{1}{2}\frac{t^2}{{h_0}^2} h\left(g_1^h + g_2^h\right)^2 d^{1,h}(x)
\end{aligned}
$$

Thank you!