The passage from 3d to 2d in mathematical elasticity

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Outline

Elastic Shells


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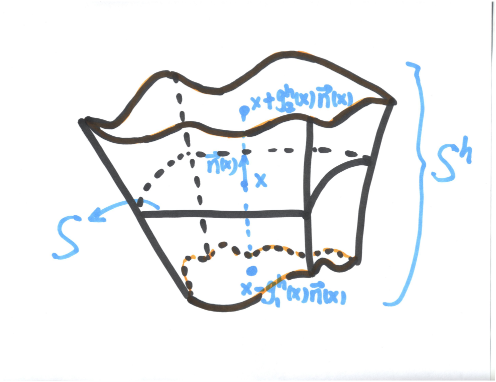
Elastic Shells

$S \subset \mathbb{R}^3$ a 2d surface imbedded in $\mathbb{R}^3$ and $\vec{n}(x)$ is its unit normal vector at $x \in S$.



$$S^h = \{ z = x + t\vec{n}(x); x \in S, -g_1^h(x) < t < g_2^h(x) \}$$

3d Elasticity

- The equations for the balance of linear momentum for $u = u(t,x)$:

  $$\partial_{tt} u - \mathrm{div} DW(\nabla u) = f, \quad \forall x \in \Omega \subset R^3,$$

  where
  - u is the deformation on Ω
  - DW is the Piola-Kirchhoff stress tensor, f is the external force
  - W elastic energy density, enjoying physical relevant conditions.
- The equilibrium equations:

  $$-\mathrm{div} DW(\nabla u) = f, \quad \forall x \in \Omega \subset \mathbb{R}^3.$$

- Total energy

  $$J^h(u) = \int_\Omega W(\nabla u) + \int_\Omega f \cdot u.$$

3d Elasticity

- Total energy for shells

$$J^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h) + \frac{1}{h}\int_{S^h} f^h \cdot u^h$$

- Elastic energy

$$I^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h)$$

- Scaling of $I^h(u^h)$ at the minimizers of J^h:

$$f^h \sim h^\alpha \Longrightarrow I^h(u^h) \sim h^\beta.$$

  - $\beta = \alpha$, if $0 \leq \alpha \leq 2$.
  - $\beta = 2\alpha - 2$, if $\alpha \geq 2$.

Examples of 2d theories and formal justification

- Examples of 2d theories
  - Membrane theory
  - Bending theory · Kirchhoff theory · Flexural theory
  - von Kármán theory
- Formal justification
  - Main method: Formal asymptotic expansion (Introduced by Ciarlet, Destuynder)
  - Ciarlet, Destuynder, Lods, Fox, Raoult, Simo, Miara and etc.
- Rigorous justification
  - Main method: Γ-convergence (Introduced by De Giorgi and Franzoni)
  - LeDret, Raoult, Friesecke, James, Müller, Conti, Maggi, Lewicka, Mora, Pakzad and etc.

Γ-convergence

Definition of Γ-convergence

$\mathcal{F}_n, \mathcal{F} : X \longrightarrow \bar{\mathbb{R}}, \mathcal{F}_n \xrightarrow{\Gamma} \mathcal{F}$ iff:

(i) Lower Bound For each $x_n \rightarrow x$ in X: $\liminf_{n\rightarrow\infty} \mathcal{F}_n(x_n) \geq \mathcal{F}(x)$.

(ii) Recovery Sequence For every $x \in X$, there exists $x_n \rightarrow x$, such that: $\lim_{n\rightarrow\infty} \mathcal{F}_n(x_n) = \mathcal{F}(x)$.

Metatheorem

$\mathcal{F}^h \xrightarrow{\Gamma} \mathcal{F} + Compactness \Longrightarrow$ The global minimizers of $\mathcal{F}^h$ converge to the global minimizers of $\mathcal{F}$

Remark: $\mathcal{F}$ is lower semi-continuous.

Metatheorem in Details

Assume $F_n \xrightarrow{\Gamma} F$, then we have the following two properties.

1. $3d \longrightarrow 2d$:

$$\left.\begin{array}{c} \lim\limits_{n\to\infty}\left\{F_n(x_n) - \inf\limits_X F_n\right\} = 0 \\ x_n \longrightarrow x \end{array}\right\} \Rightarrow F(x) \le F(y), \quad \forall y \in X.$$

2. $2d \longrightarrow 3d$:

$$\left.\begin{array}{c} F(x) \le F(y),\ \forall\ y \in X \\ x_n \longrightarrow x,\ \lim\limits_{n\to\infty} F_n(x_n) = F(x) \\ F_n(y_n)\ \text{ is bounded} \Rightarrow y_{n_k} \longrightarrow y_0 \in X \end{array}\right\} \Rightarrow \lim_{n\to\infty}\left\{F_n(x_n) - \inf_X F_n\right\} = 0.$$

Rigorous Justifications

- Plate theories, $S \subset \mathbb{R}^2$, $g_1^h(x) = g_2^h(x) = h/2$
    - $\beta = 0$ · Membrane theory · LeDret and Raoult (1995)
    - $\beta \geq 2$ · Bending theory and von Kármán theory · Friesecke, James and Müller (2002, 2006)
    - $0 < \beta < 5/3$ · Conti and Maggi (2008)
    - $5/3 \leq \beta < 2$ Open, crumpling of elastic sheets
- Shell theories, general $S \subset \mathbb{R}^3$, $g_1^h(x) = g_2^h(x) = h/2$
    - $\beta = 0$ · Membrane theory · LeDret and Roult (1996)
    - $\beta = 2$ · Bending theory · Friesecke, James, Mora and Müller (2003)
    - $\beta \geq 4$ Lewicka, Mora and Pakzad (2009)
- $g_1^h = hg_1, g_2^h = hg_2$, $\beta = 4$ · von Kármán theory · Lewicka, Mora and Pakzad (2009)

von Kármán Theory for varying thickness Shells

$$S^h = \left\{ z = x + t\vec{n}; x \in S,\ t \in (-g_1^h, g_2^h) \right\},$$

- g_1^h, g_2^h are $\mathcal{C}^1$ s.t. $\exists$ $\mathcal{C}^1$ functions $g_1, g_2 : S \longrightarrow \mathbb{R}_+$ s.t.

$$\frac{g_i^h}{h} \to g_i \quad \text{in} \quad \mathcal{C}^1(S) \quad \textit{for} \quad i = 1, 2.$$

Theorem 1 (Compactness and lower bound)

Let a sequence of deformations $u^h \in W^{1,2}(S^h)$ satisfying:

$$I^h(u^h) \leq Ch^4.$$

Then $\exists Q^h \in SO(3)$ and $\exists c^h \in \mathbb{R}^3$ such that for $\tilde{u}^h(z) = (Q^h)^T u^h(z) + c^h$ the following hold:

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(i) $(\tilde{u}^h \circ s^h)(x + t\vec{n}) \to x$ in $W^{1,2}(S^*)$, as $h \to 0$,

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(i) $(\tilde{u}^h \circ s^h)(x + t\vec{n}) \to x$ in $W^{1,2}(S^*)$, as $h \to 0$,

(ii) $V^h \to V \in \mathcal{V}$ in $W^{1,2}(S)$, as $h \to 0$,

- $V^h = \dfrac{1}{h} \displaystyle\int_{-h_0/2}^{h_0/2} (\tilde{u}^h - id) \circ s^h(x + t\vec{n}(x))\mathrm{d}t,$
- $\mathcal{V}$ consists of all first order isometries, i.e.:

for each $V \in \mathcal{V}$, $\mathrm{sym}\nabla V = 0$

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(ii) $V^h \to V \in \mathcal{V}$ in $W^{1,2}(S)$, as $h \to 0$,

(iii) $\frac{1}{h}\text{sym}\nabla V^h \rightharpoonup B_{tan} \in \mathcal{B}$ in $L^2(S)$,as $h \to 0$,

- the space of finite strains $\mathcal{B} = cl_{L^2}\{\text{sym}\nabla w; w \in W^{1,2}(S, \mathbb{R}^3)\}$.

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- (iii) $\frac{1}{h}\mathrm{sym}\nabla V^h \rightharpoonup B_{tan} \in \mathcal{B}$ in $L^2(S)$,as $h \to 0$,
- (iv) $\liminf\limits_{h\to 0} \frac{1}{h^4} I^h(u^h) \geq I(V, B_{tan})$

The generalized von Kármán functional on S

$$I(V,B_{tan}) = \frac{1}{2}\int_S (g_1+g_2)\mathcal{Q}_2\left(x, B_{tan} - \frac{1}{2}(A^2)_{tan} - \frac{1}{2}\mathrm{sym}\left(A\nabla((g_2-g_1)\vec{n})\right)\right) + \frac{1}{24}\int_S (g_1+g_2)^3\ \mathcal{Q}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{tan}\right)$$

where

$$\mathcal{Q}_2(x, F_{tan}) = min\{\mathcal{Q}_3(\tilde{F}); (\tilde{F}-F)_{tan} = 0\}, \mathcal{Q}_3(F) = D^2W(\mathrm{Id})(F,F)$$

- Stretching: $B_{tan} - \frac{1}{2}(A^2)_{tan} - \frac{1}{2}\mathrm{sym}\left(A\nabla((g_2-g_1)\vec{n})\right)$
- Bending: $(\nabla(A\vec{n}) - A\Pi)_{tan}$

von Kármán Theory for varying thickness Shells

Theorem 2 (Recovery Sequence)

For each $V \in \mathcal{V}$, $B_{tan} \in \mathcal{B}$, there exists $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ such that:

(i) $(u^h \circ s^h)(x + t\vec{n}) \to x$ in $W^{1,2}(S^*)$, as $h \to 0$,

(ii) $V^h \to V \in \mathcal{V}$ in $W^{1,2}(S)$, as $h \to 0$,

(iii) $\frac{1}{h}\mathrm{sym}\nabla V^h \to B_{tan} \in \mathcal{B}$ in $L^2(S)$, as $h \to 0$,

(iv) $\lim\limits_{h\to 0} \dfrac{1}{h^4} I^h(u^h) = I(V, B_{tan})$.

where the definition for each quantity is exactly the same as in previous theorem, with $\tilde{u}^h = u^h$.

Key point of the proof

Rigidity Estimate (Friesecke, James and Müller 2002)

$\forall u \in W^{1,2}(\Omega, \mathbb{R}^3), \exists R \in SO(3)$ such that

$$\int_\Omega |\nabla u - R|^2 \leq C \int_\Omega \mathrm{dist}^2(\nabla u, SO(3))$$

and $C = C(\Omega)$.

Remark: $C(S^h)$ is of order h^{-2}.
Based on the Rigidity Estimate, we obtain compactness of the sequence u^h with controlled energy. The lower bound by $I(V, B_{tan})$ follows through formal Taylor expansion of $W(\nabla u)$ around Id.

More on Rigidity Theorem

- Liouville Theorem: If a smooth mapping $v : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, satisfies $\nabla v \in SO(3)$, then it is affine, i.e. there exist $R \in SO(n)$ and $c \in \mathbb{R}^n$ such that $v(x) = Rx + c$.
- Korn's inequality: Assume $\Omega \subset \mathbb{R}^n$ is an open bounded domain with Lipschitz boundary, then for every $u \in W^{1,2}(\Omega, \mathbb{R}^n)$, there exists a skew symmetric matrix A and a vector $b \in \mathbb{R}^n$, such that

$$\|u - (Ax + b)\|_{W^{1,2}(\Omega)} \leq C_\Omega \|\mathrm{sym}\nabla u\|_{L^2(\Omega)}.$$

- Poincaré's inequality: Let $\Omega \subset \mathbb{R}^n$ bounded, connected open with C^1 boundary. Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p and Ω, such that

$$\left\| u - \fint_\Omega u dy \right\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

for each $u \in W^{1,p}(\Omega)$.

Thank you!