Metric Induced Shape of a Thin Film

Shape Formation in Flowers and Leaves

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Outline

1. Mechanism
2. Mathematical Model
3. Γ-convergence
4. Main Result
5. Connection to Elasticity
6. Open Problems


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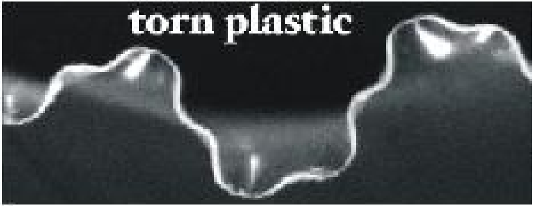
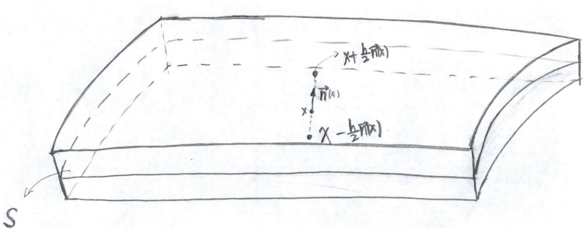
Mechanism



The thin film (petal) attempts to realize a growth-imposed non-Euclidean metric

Mathematical Model

$$S^h = \left\{ z = x + t\vec{n}(x); x \in S, t \in \left(-\frac{h}{2}, \frac{h}{2} \right) \right\}$$



$u^h : S^h \longrightarrow \mathbb{R}^3$; g^h: metric on S^h

Realization of g^h by deformation u^h in $\mathbb{R}^3$

- Orientation preserving isometric immersion of g^h into $\mathbb{R}^3$ with Euclidean metric

$$(\nabla u^h)^T \nabla u^h = g^h \quad \text{and} \quad \det \nabla u^h > 0$$

$$\Updownarrow$$

$$\nabla u^h \left(\sqrt{g^h}\right)^{-1} \in SO(3).$$

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Q 3: How to measure the closeness of a deformation to a given metric?

Closeness of a deformation to g^h

- Deformation u^h realizes g^h iff

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- Minimizer?

Elastic Energy

- Elastic energy per unit thickness

$$I^h(u^h) = \frac{1}{h}\int_{S^h} W\left(\nabla u^h \left(\sqrt{g^h}\right)^{-1}\right) \mathrm{d}z$$

  - Energy density functional $W \sim \mathrm{dist}^2(\cdot, \mathrm{SO}(3))$.
  - $I^h(u^h) \sim h^\beta$.

- Thin 3D → 2D??

Γ-convergence

One of the main tools in **Calculus of Variations**

Definition of Γ-convergence

$F_n, \quad F : X \longrightarrow \bar{\mathbb{R}}$, $F_n \xrightarrow{\Gamma} F$ iff:

(i) For $\forall\ x_n \to x$ in X, $\liminf\limits_{n\to\infty} F_n(x_n) \geq F(x)$. Lower Bound

(ii) $\forall\ x \in X$, $\exists\ x_n \to x$, s.t. $\lim\limits_{n\to\infty} F_n(x_n) = F(x)$. Recovery Sequence

Detailed properties about Γ-convergence

Assume $F_n \xrightarrow{\Gamma} F$, then we have the following two properties.

1. $3d \longrightarrow 2d$:

$$\left.\begin{array}{c} \lim\limits_{n\to\infty}\left\{F_n(x_n) - \inf\limits_X F_n\right\} = 0 \\ x_n \longrightarrow x \end{array}\right\} \Rightarrow F(x) \leq F(y), \quad \forall y \in X.$$

2. $2d \longrightarrow 3d$:

$$\left.\begin{array}{c} F(x) \leq F(y), \ \forall\ y \in X \\ x_n \longrightarrow x, \ \lim\limits_{n\to\infty} F_n(x_n) = F(x) \\ F_n(y_n) \ \text{ is bounded} \Rightarrow y_{n_k} \longrightarrow y_0 \in X \end{array}\right\} \Rightarrow \lim_{n\to\infty}\left\{F_n(x_n) - \inf_X F_n\right\} = 0.$$

Main Result

Assume for $\forall x \in S,\ t \in \left(-\frac{h}{2}, \frac{h}{2}\right)$,

$$g^h(x + t\vec{n}(x)) = g(x + t\vec{n}(x)) = \begin{bmatrix} [g_{\alpha\beta}(x)] & 0 \\ 0 & 1 \end{bmatrix},$$

where $g_{\alpha\beta}$ is a smooth metric on S.

Main Result

(i) **Compactness and Lower Bound**

Suppose $u^h \in W^{1,2}(S^h;\mathbb{R}^3)$ with

$$I^h(u^h) \leq Ch^2.$$

Then, $\exists$ $c^h \in \mathbb{R}^3$ s.t. $y^h(x+t\vec{n}(x)) = u^h(x+t\frac{h}{h_0}\vec{n}(x)) - c^h$ satisfies,

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(α) $y^h(x+t\vec{n}(x)) \rightarrow y(x)$, up to a subsequence in $W^{1,2}(S^{h_0},\mathbb{R}^3)$ and $y \in W^{2,2}(S,\mathbb{R}^3)$, realizing the metric $g_{\alpha\beta}$, i.e. $(\nabla y)^T\nabla y = g_{\alpha\beta}$.

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(β) The lower bound of the energy functional,

$$\liminf_{h\to 0}\frac{1}{h^2}I^h(u^h) \geq \frac{1}{24}\int_S Q_2\left(x, \left(\sqrt{g_{\alpha\beta}}\right)^{-1}\left((\nabla y)^T\nabla\vec{N} - g_{\alpha\beta}\nabla\vec{n}\right)\left(\sqrt{g_{\alpha\beta}}\right)^{-1}\right)\mathrm{d}x.$$

Limiting Energy Functional

$$\frac{1}{24}\int_S \mathcal{Q}_2\left(x, \left(\sqrt{g_{\alpha\beta}}\right)^{-1}\left((\nabla y)^T \nabla \vec{N} - g_{\alpha\beta}\nabla \vec{n}\right)\left(\sqrt{g_{\alpha\beta}}\right)^{-1}\right) \mathrm{d}x$$

where

$$\mathcal{Q}_2(x, F_{tan}) = \min\left\{\nabla^2 W(\mathrm{Id})(\tilde{F}, \tilde{F}); \tilde{F}_{tan} = F_{tan}\right\}$$

$\vec{N}$: the normal of $y(S)$.

Main Result

(ii) Recovery Sequence

For $\forall\ y \in W^{2,2}(S,\mathbb{R}^3)$, with $(\nabla y)^T \nabla y = g_{\alpha\beta}$
$\exists\ u^h \in W^{1,2}(S^h,\mathbb{R}^3)$ s.t.

$$u^h\left(x + t\frac{h}{h_0}\vec{n}(x)\right) \to y(x) \quad \text{in} \quad W^{1,2}(S^{h_0},\mathbb{R}^3),$$

and

$$\liminf_{h\to 0} \frac{1}{h^2} I^h(u^h)$$
$$= \frac{1}{24}\int_S Q_2\left(x, \left(\sqrt{g_{\alpha\beta}}\right)^{-1}\left((\nabla y)^T \nabla \vec{N} - g_{\alpha\beta}\nabla\vec{n}\right)\left(\sqrt{g_{\alpha\beta}}\right)^{-1}\right) \mathrm{d}x.$$

Construction of Recovery Sequence

$$u^h(x + t\vec{n}(x)) = y(x) + t\vec{N}(x) + \frac{1}{2}t^2 d^h(x).$$

Key Point of proof of (i)

Rigidity Estimate (Friesecke, James and Müller 2002)

$\Omega \subset \mathbb{R}^n$, open bounded with Lipschitz boundary, then $\exists C = C(\Omega)$ satisfying $\forall u \in W^{1,2}(\Omega, \mathbb{R}^3), \exists R \in SO(3)$ s.t.

$$\|\nabla u - R\|_{L^2(\Omega)} \leq C \, \|\mathrm{dist}(\nabla u, SO(3))\|_{L^2(\Omega)} .$$

Based on the Rigidity Estimate, we obtain compactness of the sequence u^h with controlled energy. The lower bound follows through formal Taylor expansion of $W(\nabla u^h (\sqrt{g})^{-1})$ around Id.

Connection to Elasticity of plates and shells

- Total energy functional

$$J^h(u^h) = I^h(u^h) - \frac{1}{h}\int_{S^h} f^h \cdot u^h$$

  - $I^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h)dz$
  - $f^h \sim h^\alpha \Longrightarrow I^h(u^h) \sim h^\beta$ at the minimizer of J^h.
    - $\beta = \alpha$, if $0 \leq \alpha \leq 2$.
    - $\beta = 2\alpha - 2$, if $\alpha \geq 2$.
- Main Question: Identify Γ-limit J of J^h or Γ-limit J_β of $h^{-\beta}J^h$. (argmin $J^h \rightarrow$ argmin J)

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- Previous Results
  (LeDret, Raoult, Friesecke, James, Müller, Conti, Maggi, Dolzmann, Mora, Lewicka, Pakzad etc.)
  - For Plates, $0 \le \beta < 5/3$ and $\beta \ge 2$
  - For Shells, $\beta = 0$, $\beta = 2$ and $\beta \ge 4$

Open Problems

- Hierarchy of non-Euclidean elastic theories.
- Varying thickness

  $$\left(-\frac{h}{2},\frac{h}{2}\right)\longrightarrow(-g_1^h,g_2^h)$$

  for some general function g_1^h, g_2^h.
- Incompressible case

  $$\det\left(\nabla u^h\right)=\det\left(\sqrt{g^h}\right).$$

Thank you!