Probabilities in the Game "13 Poker Cards"

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Abstract

In this arcitle I tried to determine the most reasonable ordering of poker cards in the game "13 Poker Cards", by calculating the probability of each type of poker cards. The calculation is a little complex, so I have used mathematical software Maxima and C program.

1 Introduction

In the game "13 Poker Cards", 52 poker cards are randomly distributed to 4 people playing this game, each one having 13 poker cards. Then each person divides his/her 13 poker cards into 3 groups, with 3, 5 and 5 poker cards in each group, with each one's 1st group smaller than the 2nd group, and 2nd group smaller than 3rd group. Then the 4 people compare their cards in each group to determine who wins the most rounds.

In this small article, I tried to determine what's the most reasonable ordering of poker cards, which is the same as solving the following math problem: "Given one deck of poker cards (52 cards in total), randomly choose 13 cards out of the the 52 cards. What's the probability of having at least one "Tong Hua Shun", "Full House", "Shun" etc.?" This small article describes how I used mathematical software such as Maxima and simple C program to calculate these probabilities.

2 Calculation of the probabilities

The order of the poker cards are defined to be (in ascending order):

$$2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$$

1. **Tong Hua Shun** ($P = \frac{9602547420}{635013559600} \approx 1.5122\%$)

   Def. Any five cards are called "Tong Hua Shun" if their numbers are consecutive, and their colors are the same.

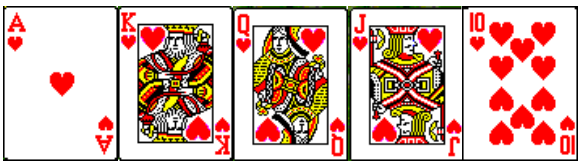

Figure 1. Tong Hua Shun

We can use two methods to calculate the probability of having at least one "Tonghua Shun" in a randomly chosen 13 poker cards.

i. The exact solution.

Let $f(x)$ be the number of instances that there're 5 or more consecutive cards in any x cards with the same color. For example, $f(4) = 0$, $f(5) = 9$, $f(13) = 1$ etc. I used a simple C program to get $f(x)$ for all $x = 0, 1, 2, ..., 13$.

```
#include <stdio.h>
int main ()
```

```
{
    int x2,x3,x4,x5,x6,x7,x8,x9,x10,xJ,xQ,xK,xA;
    int x=13;
    unsigned long long int s=0;

    printf("please enter total number:");
    scanf("%d",&x);
    printf("x=%d\n",x);
    s=0;
    for (x2=0;x2<=1;x2++)
     for (x3=0;x3<=1;x3++)
      for (x4=0;x4<=1;x4++)
       for (x5=0;x5<=1;x5++)
        for (x6=0;x6<=1;x6++)
         for (x7=0;x7<=1;x7++)
          for (x8=0;x8<=1;x8++)
           for (x9=0;x9<=1;x9++)
            for (x10=0;x10<=1;x10++)
             for (xJ=0;xJ<=1;xJ++)
              for (xQ=0;xQ<=1;xQ++)
               for (xK=0;xK<=1;xK++)
                for (xA=0;xA<=1;xA++)
                {
                   if (x2+x3+x4+x5+x6+x7+x8+x9+x10+xJ+xQ+xK+xA==x)
                   {
                    if ((x2*x3*x4*x5*x6>0) || (x3*x4*x5*x6*x7>0) ||\
                       (x4*x5*x6*x7*x8>0) || (x5*x6*x7*x8*x9>0) ||\
                       (x6*x7*x8*x9*x10>0) || (x7*x8*x9*x10*xJ>0) ||\
                       (x8*x9*x10*xJ*xQ>0) || (x9*x10*xJ*xQ*xK>0) ||\
                       (x10*xJ*xQ*xK*xA>0))
                      {
                        s=s+1;
                      }
                    }
                }
    printf("total possibilities of having 'Shun' in %d number of same
color poker cards is %llu\n",x,s);
    return 0;
}
```

There're only two possibilities:

– There is "Tonghua Shun" for exactly two colors.

To choose those two colors, there are $C_4^2 = 6$ choices. Let w, x, y, z be the number of poker cards for each color. Then

$$\begin{cases} w+x+y+z=13 \\ 5 \le w \le 8 \\ 5 \le x \le 8 \\ 0 \le y \le 4 \\ 0 \le z \le 4 \end{cases}$$

For each solution, the number of instances is

$$f(w) \cdot f(x) \cdot C_{13}^{y} \cdot C_{13}^{z}$$

– There is "Tonghua Shun" for exactly one color.

To choose that one color, there are $C_4^1 = 4$ choices. Let w, x, y, z be the number of poker cards for each color. Then

$$\begin{cases} w+x+y+z=13 \\ 5 \le w \le 13 \\ 0 \le x \le 13 \\ 0 \le y \le 13 \\ 0 \le z \le 13 \end{cases}$$

For each solution, the number of instances is

$$f(w) \cdot (f(x) - C_{13}^x) \cdot (f(y) - C_{13}^y) \cdot (f(z) - C_{13}^z)$$

Now we use Maxima to solve this problem.

```
Maxima restarted.
(%i1) f(x):=block([x:x],
             if x<0 or x>13 then return(0),
             if 0<=x and x<=4 then return(0),
             if x=5 then return(9),
             if x=6 then return(64),
             if x=7 then return(196),
             if x=8 then return(336),
             if x=9 then return(350),
             if x=10 then return(218),
             if x=11 then return(75),
             if x=12 then return(13),
             if x=13 then return(1));
```

(%o1) $f(x) := \textbf{block}\ ([x: x], \textbf{if}\ x < 0 () x > 13\ \textbf{then}\ \text{mreturn}(0), \textbf{if}\ 0 \le x () x \le 4\ \textbf{then}\ \text{mreturn}(0), \textbf{if}\ x = 5\ \textbf{then}\ \text{mreturn}(9), \textbf{if}\ x = 6\ \textbf{then}\ \text{mreturn}(64), \textbf{if}\ x = 7\ \textbf{then}\ \text{mreturn}(196), \textbf{if}\ x = 8\ \textbf{then}\ \text{mreturn}(336), \textbf{if}\ x = 9\ \textbf{then}\ \text{mreturn}(350), \textbf{if}\ x = 10\ \textbf{then}\ \text{mreturn}(218), \textbf{if}\ x = 11\ \textbf{then}\ \text{mreturn}(75), \textbf{if}\ x = 12\ \textbf{then}\ \text{mreturn}(13), \textbf{if}\ x = 13\ \textbf{then}\ \text{mreturn}(1))$

```
(%i2) f(5);
```

(%o2) 9

```
(%i3) s:0;
```

(%o3) 0

```
(%i4) for w:5 thru 8 step 1 do
       for x:5 thru 8 step 1 do
        for y:0 thru 4 step 1 do
         for z:0 thru 4 step 1 do
          if w+x+y+z=13 then
           s:s+6*f(w)*f(x)*binomial(13,y)*binomial(13,z);
```

(%o4) done

```
(%i5) s;
```

(%o5) 4886160

```
(%i12) for w:5 thru 13 step 1 do
        for x:0 thru 13 step 1 do
         for y:0 thru 13 step 1 do
          for z:0 thru 13 step 1 do
           if w+x+y+z=13 then
            s:s+4*f(w)*(binomial(13,x)-f(x))*(binomial(13,y)-f(y))
                 *(binomial(13,z)-f(z));
```

```
(%o6) done
(%i7) s;

(%o7) 9602547420
(%i8) s/binomial(52,13);
```

$$(\%o8)\ \frac{480127371}{31750677980}$$

```
(%i9) float(%);

(%o9) 0.01512179901489
```

ii. Approximate Solution (Thanks to Health@mitbbs.com).

$$\frac{9\cdot 4\cdot C_{52-5}^{13-5}-8\cdot 4\cdot C_{52-6}^{13-6}}{C_{52}^{13}}\ \approx\ 0.01512988173993$$

2. **Tie Zhi** ($P=\frac{21717689136}{635013559600}\approx 3.420\%$)
Def. Any 5 cards are called "Tie Zhi" if 4 of them are of the same number.

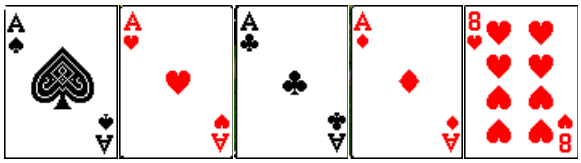

Figure 2. Tie Zhi

```
Maxima restarted.
(%i1) s:0$
(%i44) for w:0 thru 3 step 1 do
          for x:0 thru 4 step 1 do
              for y:0 thru 6 step 1 do
                   for z:0 thru 13 step 1 do
                      if 4*w+3*x+2*y+z=13 and w>0 then
                         s:s+binomial(13,w)*binomial(13-w,x)*binomial(13-
      w-x,y)*binomial(13-w-x-y,z)*4^x*6^y*4^z;

(%o44) done
(%i45) s;

(%o45) 21717689136
(%i46) binomial(52,13);

(%o46) 635013559600
(%i47) s:s/binomial(52,13);
```

$$(\%o47)\ \frac{1357355571}{39688347475}$$

```
(%i48) float(%);
```

```
(%o48) 0.0342003549494
```

3. **Full House** ($P = \frac{281333546208}{635013559600} \approx 44.304\%$)

   Def. Any 5 poker cards areeee called "Full House" if 3 of them are of the same number and the other 2 are of the same another number.

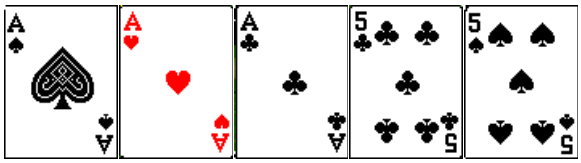

Figure 3. Full House

```
Maxima restarted.

(%i1) s:0$
(%i2) for w:0 thru 3 step 1 do
          for x:0 thru 4 step 1 do
              for y:0 thru 6 step 1 do
                   for z:0 thru 13 step 1 do
                      if 4*w+3*x+2*y+z=13 and x>0 and y>0 then
                         s:s+binomial(13,w)*binomial(13-w,x)*binomial(13-w-
      x,y)*binomial(13-w-x-y,z)*4^x*6^y*4^z;

(%o2) done
(%i3) s;

(%o3) 281333546208
(%i4) binomial(52,13);

(%o4) 635013559600
(%i5) s:s/binomial(52,13);

                          17583346638
(%o5)                     -----------
                          39688347475
(%i6) float(%);

(%o46) 0.44303549421089
```

4. **San Tiao** ($P = \frac{293959370848}{635013559600} \approx 46.292\%$)

   Def. Any 5 poker cards are called "San Tiao" if 3 of them are of the same number. Full house is a special case of San Tiao.

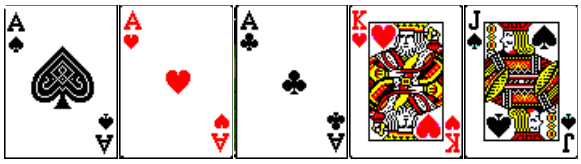

Figure 4. San Tiao

```
Maxima restarted.

(%i1) s:0$
```

```
(%i20) for w:0 thru 3 step 1 do
           for x:0 thru 4 step 1 do
               for y:0 thru 6 step 1 do
                    for z:0 thru 13 step 1 do
                       if 4*w+3*x+2*y+z=13 and x>0 then
                          s:s+binomial(13,w)*binomial(13-w,x)*binomial(13-
       w-x,y)*binomial(13-w-x-y,z)*4^x*6^y*4^z;

(%o20) done
(%i21) s;

(%o21) 293959370848
(%i22) binomial(52,13);

(%o22) 635013559600
(%i23) s:s/binomial(52,13);

                  18372460678
(%o23)            -----------
                  39688347475
(%i24) float(%);

(%o24) 0.46291825804975
(%i25)
```

5. **Shun** ($P=\frac{329191202528}{635013559600}\approx 51.840\%$)

   Def. Any 5 cards are called "Shun" if their numbers are consecutive.

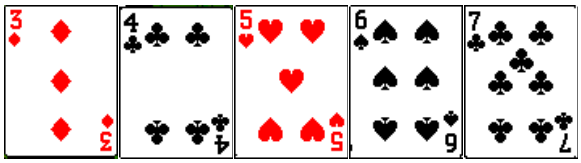

Figure 5. Shun

```
#include <stdio.h>
int main ()
{
    int x2,x3,x4,x5,x6,x7,x8,x9,x10,xJ,xQ,xK,xA;
    int x=13;
    unsigned long long int s=0;

    printf("please enter total number:");
    scanf("%d",&x);
    printf("x=%d\n",x);
    s=0;
    for (x2=0;x2<=4;x2++)
     for (x3=0;x3<=4;x3++)
      for (x4=0;x4<=4;x4++)
       for (x5=0;x5<=4;x5++)
        for (x6=0;x6<=4;x6++)
         for (x7=0;x7<=4;x7++)
          for (x8=0;x8<=4;x8++)
           for (x9=0;x9<=4;x9++)
```

```
                for (x10=0;x10<=4;x10++)
                 for (xJ=0;xJ<=4;xJ++)
                  for (xQ=0;xQ<=4;xQ++)
                   for (xK=0;xK<=4;xK++)
                    for (xA=0;xA<=4;xA++)
                    {
                        if (x2+x3+x4+x5+x6+x7+x8+x9+x10+xJ+xQ+xK+xA==x)
                        {
                            if ((x2*x3*x4*x5*x6>0) || (x3*x4*x5*x6*x7>0) ||\
                                (x4*x5*x6*x7*x8>0) || (x5*x6*x7*x8*x9>0) ||\
                                (x6*x7*x8*x9*x10>0) || (x7*x8*x9*x10*xJ>0) ||\
                                (x8*x9*x10*xJ*xQ>0) || (x9*x10*xJ*xQ*xK>0) ||\
                                (x10*xJ*xQ*xK*xA>0))
                            {
                             s=s+binom4(x2)*binom4(x3)*binom4(x4)*\
                                 binom4(x5)*binom4(x6)*binom4(x7)*\
                                 binom4(x8)*binom4(x9)*binom4(x10)*\
                                 binom4(xJ)*binom4(xQ)*binom4(xK)*binom4(xA);
                            }
                        }
                    }
    printf("total possibilities of having 'Shun' in %d number of poker
cards is %llu\n",x,s);
    return 0;
}

int binom4(int x)
{
    int r;
    switch(x)
    {
        case 0:
            r=1;
            break;
        case 1:
            r=4;
            break;
        case 2:
            r=6;
            break;
        case 3:
            r=4;
            break;
        case 4:
            r=1;
            break;
        default:
            r=0;
    }
    return r;
}
```

Total possibilities of having 'Shun' in 13 number of poker cards is 329191202528

$$\frac{329191202528}{C_{52}^{13}} \approx 0.51840027279947$$

6. **Tong Hua** ($P = \frac{412247470340}{635013559600} \approx 64.920\%$)

Def. Any 5 poker cards are called "Tong Hua", if their colors are the same.

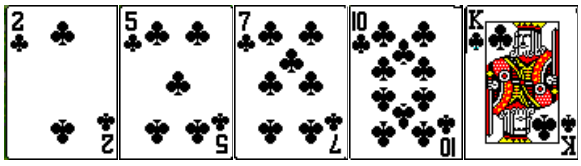

Figure 6. Tong Hua

```
(%i1) s:0;
```

```
(%o10) 0
```

```
(%i11) for w:0 thru 13 step 1 do
         for x:0 thru 13 step 1 do
           for y:0 thru 13 step 1 do
             for z:0 thru 13 step 1 do
               if w+x+y+z=13 and (w>4 or x>4 or y>4 or z>4) then

       s:s+binomial(13,w)*binomial(13,x)*binomial(13,y)*binomial(13,z);
```

```
(%o11) done
```

```
(%i12) s;
```

```
(%o12) 412247470340
```

```
(%i13) s:s/binomial(52,13);
```

$$\text{(\%o13)}\quad \frac{20612373517}{31750677980}$$

```
(%i14) float(%);
```

```
(%o14) 0.64919475199817
```

7. **Two Pair** ($P=\frac{547890973344}{635013559600}\approx 86.280\%$)

Def. Any 5 poker cards are called "Two Pair", if there are exactly two pairs of poker cards in them, each pair having the same number.

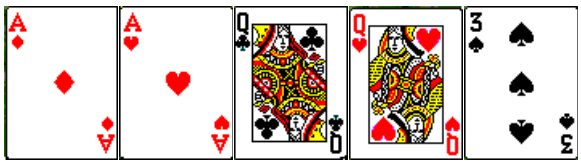

Figure 7. Two Pair

```
Maxima restarted.
```

```
(%i1) s:0$
(%i2) for w:0 thru 3 step 1 do
        for x:0 thru 4 step 1 do
          for y:0 thru 6 step 1 do
            for z:0 thru 13 step 1 do
              if 4*w+3*x+2*y+z=13 and y>1 then
                s:s+binomial(13,w)*binomial(13-w,x)*binomial(13-w-
    x,y)*binomial(13-w-x-y,z)*4^x*6^y*4^z;
```

```
(%o2) done
(%i3) s;

(%o3) 547890973344
(%i4) binomial(52,13);

(%o4) 635013559600
(%i5) s:s/binomial(52,13);

(%o5) 728578422/844432925
(%i6) float(%);

(%o6) 0.86280200644711
(%i7)
```

8. **One Pair** ($P = \frac{621533981952}{635013559600} \approx 97.877\%$)

Def. Any 5 poker cards are called "One Pair", if there are at least one pair of poker cards in them, having the same number.

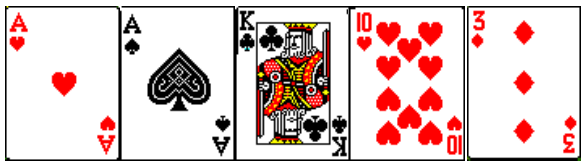

Figure 8. One Pair.

```
Maxima restarted.
(%i1) s:0$
(%i8) for w:0 thru 3 step 1 do
          for x:0 thru 4 step 1 do
              for y:0 thru 6 step 1 do
                   for z:0 thru 13 step 1 do
                      if 4*w+3*x+2*y+z=13 and y>0 then
                         s:s+binomial(13,w)*binomial(13-w,x)*binomial(13-w-
      x,y)*binomial(13-w-x-y,z)*4^x*6^y*4^z;

(%o8) done
(%i9) s;

(%o9) 621533981952
(%i10) binomial(52,13);

(%o10) 635013559600
(%i11) s:s/binomial(52,13);

(%o11) 38845873872/39688347475
(%i12) float(%);
```

```
(%o12) 0.97877277194444
(%i13)
```

3 Conclusion

By the above calculation, the most reasonable ordering of poker cards are (in descending order):

1. Tong Hua Shun $(P = \frac{9602547420}{635013559600} \approx 1.5122\%)$
2. Tie Zhi $(P = \frac{21717689136}{635013559600} \approx 3.420\%)$
3. Full House $(P = \frac{281333546208}{635013559600} \approx 44.304\%)$
4. San Tiao $(P = \frac{293959370848}{635013559600} \approx 46.292\%)$
5. Shun $(P = \frac{329191202528}{635013559600} \approx 51.840\%)$
6. Tong Hua $(P = \frac{412247470340}{635013559600} \approx 64.920\%)$
7. Two Pair $(P = \frac{547890973344}{635013559600} \approx 86.280\%)$
8. One Pair $(P = \frac{621533981952}{635013559600} \approx 97.877\%)$