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EXISTENCE OF ENERGY MINIMIZERS FOR MAGNETOSTRICTIVE MATERIALS

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Abstract. The existence of a deformation and magnetization minimizing the magnetostrictive free energy is given. Mathematical challenges are presented by a free energy that includes elastic contributions defined in the reference configuration and magnetic contributions defined in the spatial frame. The one-to-one a.e. and orientation-preserving property of the deformation is demonstrated, and the satisfaction of the nonconvex saturation constraint for the magnetization is proven.

1. Introduction

A mathematical model for magnetostrictive materials, in which the deformation and the magnetization are coupled, has been given in [4, 8, 19, 20, 22]. More significant shape change can be obtained from magnetostrictive materials that also undergo a structural phase transformation since the material can then lower its energy by an increase in the volume fraction of the martensitic variant with magnetic easy axis aligned with the applied magnetic field [11, 21]. This shape change is being utilized in emerging applications in actuators, sensors, and micromachines because they can provide a large *work output*/(*cycle* · *volume*) [25].

In this paper, we prove the existence of a deformation and magnetization minimizing the magnetostrictive free energy [4, 19, 20]. A novel feature of this free energy is that the elastic free energy is given in a reference configuration, as usual for elasticity, but the magnetic energies are given in the spatial frame. This requires that we use a free energy for which the Jacobian of the deformation can be controlled. We also must prove that the deformation is one-to-one a.e. and that the magnetization satisfies the nonconvex saturation constraint.

The existence of low energy phase and variant interfaces in single crystal thin films that do not exist in bulk martensitic crystals offers the possibility to develop thin materials with substantially larger strains than are possible in bulk [2]. Magnetostrictive, shape memory, single crystal, thin films such as Ni_2MnGa have recently been grown [11, 21]. We hope that this work will provide the foundation for our future work on the derivation of a rigorous magnetostrictive thin film energy to extend the results developed for micromagnetics [17] and martensitic deformation [2, 5–7].

The plan of our work is the following. In §2, we introduce the magnetostriction model and in particular we present the constitutive assumptions. We shall seek the minimizers in an admissible set $\mathcal{A}$, described in §2, that incorporates all of the constraints. To prove the existence of minimizers, we demonstrate in Theorem 3.1 that $\mathcal{A}$ is closed in a weak topology. We then show in §4 that $\mathcal{E}$ is lower semicontinuous on $\mathcal{A}$, and hence that it attains its minimum. This is the content of Theorem 4.1 and is the main result of this paper.

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2. The magnetostriction model

We consider a magnetostrictive crystal which in an undeformed state occupies the Lipschitz domain $\Omega \subset \mathbb{R}^3$. The admissible deformations $\bar{y} : \Omega \to \mathbb{R}^3$ of the crystal will be required to have the regularity $y \in W^{2,2}(\Omega; \mathbb{R}^3)$, to be orientation-preserving ($\det \nabla y > 0$ a.e. [18]), and to be one-to-one a.e., where we recall that a mapping $y : \Omega \to \mathbb{R}^3$ is *one-to-one a.e.* if there is a set $G \subset \Omega$ of full measure (that is, $|G| = |\Omega|$) such that $y(x)$ restricted to G is one-to-one. The one-to-one property everywhere of $y(x)$ would seem more appropriate, however it is the weaker property that we will prove is preserved under weak convergence in $W^{2,2}(\Omega; \mathbb{R}^3)$.

The demonstration that energy-minimizing deformations obtained from the calculus of variations are one-to-one often requires sophisticated techniques [28], such as for problems with cavitation [27] or other problems with low regularity. Since our problem has higher regularity, simpler methods based on Banach's indicatrix are sufficient.

Because of the continuity of $y \in W^{2,2}(\Omega; \mathbb{R}^3)$, it follows that the sets $y(\bar{\Omega})$ and $y(\partial\bar{\Omega})$ are closed and the deformed domain $\mathcal{O}(y) := y(\bar{\Omega}) \setminus y(\partial\bar{\Omega})$ is open. Here and in the sequel the bar over a set denotes its closure. We note that $\mathcal{O}(y)$ differs from $y(\Omega)$ on a set of measure zero and $|\mathcal{O}(y)| = |y(\bar{\Omega})|$ [14].

We wish to model a crystal that is attached on a nonempty, open subset of its boundary, $\Gamma \subset \partial\Omega$, so we will assume that admissible deformations $y(x)$ satisfy the boundary condition

$$y(x) = y_0(x) \qquad \text{for all } x \in \Gamma, \tag{2.1}$$

where $y_0 : \bar{\Omega} \to \mathbb{R}^3$ is a C^2 diffeomorphism with positive Jacobian.

The magnetization $m(z)$ of the crystal is naturally defined in spatial coordinates by $m : \mathcal{O}(y) \to \mathbb{R}^3$ and admissible magnetizations will be required to have the regularity $m \in W^{1,2}(\mathcal{O}(y); \mathbb{R}^3)$ which is equivalent to

$$\int_{\mathcal{O}(y)} \left[|\nabla_z m(z)|^2 + |m(z)|^2\right] \, dz = \int_{\Omega} \left[|\nabla_z m(y(x))|^2 + |m(y(x))|^2\right] \det \nabla y(x) \, dx < \infty.$$

We will often find it convenient as above to consider the magnetization $m \circ y : \Omega \to \mathbb{R}^3$ described in material coordinates. We will assume that the crystal is at a fixed temperature below the Curie temperature so that

$$|m(y(x))| \det \nabla y(x) = \tau, \qquad x \in \Omega, \tag{2.2}$$

where τ, the saturation magnetization, is a positive constant depending on the temperature.

The applied magnetic field will be given in spatial coordinates by $h : \mathbb{R}^3 \to \mathbb{R}^3$, and we will assume that $h \in L^2(\mathbb{R}^3; \mathbb{R}^3)$.

The free energy of a magnetostrictive crystal can be modeled by [4, 20]:

$$\begin{aligned} &\mathcal{E}(y, m) \\ &= \int_{\Omega} \left\{ \kappa \left| D^2 y(x) \right|^2 + \Phi\left(\nabla y(x), m \circ y(x)\right) \right\} dx + \int_{\mathcal{O}(y)} \left\{ \alpha \left|\nabla_z m(z)\right|^2 - h(z) \cdot m(z) \right\} dz \\ &\qquad + e_{\mathrm{mag}}(y, m) \\ &= \int_{\Omega} \left\{ \kappa \left| D^2 y(x) \right|^2 + \Phi\left(\nabla y(x), m \circ y(x)\right) + \Big(\alpha \left|\nabla_z m\left(y(x)\right)\right|^2 - h(y(x)) \cdot m(y(x)) \Big) \det \nabla y(x) \right\} dx \\ &\qquad + e_{\mathrm{mag}}(y, m). \end{aligned} \tag{2.3}$$

where the magnetostatic energy $e_{\mathrm{mag}}(y,m)$ is calculated from the magnetic scalar potential $\zeta : \mathbb{R}^3 \to \mathbb{R}$ by

$$e_{\mathrm{mag}}(y,m) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_z \zeta(z)|^2 \, dz, \tag{2.4}$$

and where the magnetic scalar potential ζ satisfies

$$\mathrm{div}_z \left(-\nabla_z \zeta + \chi_{\mathcal{O}(y)} m\right) = 0, \qquad z \in \mathbb{R}^3. \tag{2.5}$$

We use the definition

$$\int_\Omega \left|D^2 y(x)\right|^2 \, dx = \int_\Omega \left[\sum_{i,j=1}^{3} \left|\frac{\partial^2 y(x)}{\partial x_i \partial x_j}\right|^2\right] dx,$$

and we recall above that $\chi_{\mathcal{O}(y)}(z)$ is the characteristic function of $\mathcal{O}(y)$. The terms in (2.3) represent, from left to right, the surface energy [2,23,26], the anistropy energy, the exchange energy [9,17], the interaction energy due to the applied magnetic field, and the magnetostatic energy. The parameters α and κ are positive material constants depending on the fixed temperature. The anisotropy energy density $\Phi(F,m)$ is a continuous function of the deformation gradient $F \in \mathbb{R}^{3\times 3}_+$ (where $\mathbb{R}^{3\times 3}_+$ denotes the group of 3×3 matrices with positive determinant) and the magnetization $m \in \mathbb{R}^3$. (Since the temperature is assumed to be fixed, we do not explicitly denote the dependence of the anisotropy energy density $\Phi(F,m)$ on temperature.)

We will assume that the anisotropy free energy density $\Phi \in C^2(\mathbb{R}^{3\times 3}_+ \times \mathbb{R}^3; \mathbb{R})$ is of the form

$$\Phi(F,m) = W(F,m) + \psi(\det F); \tag{2.6}$$

where $\psi : (0,\infty) \to \mathbb{R}$ is continuous, convex, and satisfies for

$$q > 2 \qquad \text{and} \qquad c_L > 0$$

the growth conditions

$$c_L(a^{-q} + a^q) \le \psi(a) \quad \text{for all } 0 < a < +\infty; \tag{2.7}$$

and where $W : \mathbb{R}^{3\times 3}_+ \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies for

$$2 \le r < 6 \qquad \text{and} \qquad 0 < C_L < C_U$$

the growth conditions

$$C_L(|F|^2 - 1) \le W(F,m) \le C_U(|F|^r + 1) \quad \text{for all } F \in \mathbb{R}^{3\times 3}_+ \text{ and } \ m \in \mathbb{R}^3. \tag{2.8}$$

We shall define the set of admissible functions to be

$$\mathcal{A} = \Big\{ (y,m) \in W^{2,2}(\Omega;\mathbb{R}^3) \times W^{1,2}(\mathcal{O}(y);\mathbb{R}^3) : y(x) = y_0(x) \text{ for all } x \in \Gamma,$$
$$\psi(\det \nabla y) \in L^1(\Omega),\ \det \nabla y > 0 \text{ a.e., and } y \text{ is one-to-one a.e.} \Big\},$$

where the growth properties of ψ were given in (2.7). We note that $\mathcal{A}$ is nonempty since we assumed in (2.1) that $y_0 : \bar{\Omega} \to \mathbb{R}^3$ is a C^2 diffeomorphism with positive Jacobian. We also note that $\mathcal{A}$ is not an affine space.

We shall show under the above assumptions that the problem

$$\min\{\mathcal{E}(y,m) : (y,m) \in \mathcal{A} \text{ and } |m(y(x))| \det \nabla y(x) = \tau \text{ for almost all } x \in \Omega\} \tag{2.9}$$

has a solution. We shall see that the terms

$$\int_{\mathcal{O}(y)} |\nabla_z m(z)|^2 \, dz = \int_\Omega |\nabla_z m|^2 \det \nabla y \, dx \quad \text{ and } \quad \int_\Omega \psi(\det \nabla y) \, dx$$

in $\mathcal{E}$ require the most care in the analysis.

The anisotropy energy density for magnetostrictive crystals that undergo a structural phase transformation can have the form (2.6). To see this, we note that the anisotropy energy density for magnetostrictive crystals such as Ni_2MnGa is minimized at temperatures below the martensitic transformation on the wells [20, 21]

$$\mathcal{M} = \mathrm{SO}(3)(U_1, m_1) \cup \mathrm{SO}(3)(U_1, -m_1) \cup \cdots \cup \mathrm{SO}(3)(U_N, m_N) \cup \mathrm{SO}(3)(U_N, -m_N),$$

where $\det U_1 > 0$ and where for the symmetry group, $\mathcal{G} \subset \mathrm{SO}(3)$, of the high temperature phase we have

$$\{(U_1, m_1), \ldots, (U_N, m_N)\} = \{(QU_1Q^T, Qm_1) : Q \in \mathcal{G}\}. \tag{2.10}$$

We note that SO(3) is the group of proper rotations and

$$\mathrm{SO}(3)(U_k, \pm m_k) \equiv \{(RU_k, \pm Rm_k) : R \in \mathrm{SO}(3)\} \qquad \text{for } k = 1, \ldots, N.$$

If $W(F, m)$ satisfies the property of frame indifference

$$W(RF, Rm) = W(F, m) \qquad \text{for all } F \in \mathbb{R}^{3\times 3}_+,\ m \in \mathbb{R}^3,\ R \in \mathrm{SO}(3),$$

and the property of material symmetry for the group, $\mathcal{G}$,

$$W(FQ, m) = W(F, m) \qquad \text{for all } F \in \mathbb{R}^{3\times 3}_+,\ m \in \mathbb{R}^3,\ R \in \mathcal{G},$$

then $\Phi(F, m)$ satisfies the property of frame indifference and material symmetry by the invariance of the determinant function. If $W(F, m)$ is minimized on the wells (2.10), that is,

$$W(\hat{F}, \hat{m}) < W(F, m) \qquad \text{for all } (\hat{F}, \hat{m}) \in \mathcal{M} \text{ and } (F, m) \in \mathbb{R}^{3\times 3}_+ \times \mathbb{R}^3 \setminus \mathcal{M}$$

and $\psi(a)$ is minimized at $\det U_1$, then $\Phi(F, m)$ is also minimized on the wells (2.10).

By rescaling the crystal domain, Ω, onto a domain of unit size, we can see that the dimensionless size of the surface energy coefficient (κ)and the exchange energy coefficient (α) become large for small crystals [9, 17, 26]. For the large crystal limit, the scaled micromagnetic energy converges to a phase theory which is the relaxation of the micromagnetics energy without exchange energy ($\alpha = 0$) [9].

Since energy-minimizing solutions of (2.3) without surface energy ($\kappa = 0$) and exchange energy ($\alpha = 0$) do not generally exist, microstructures defined by minimizing sequences are studied in [20]. The existence of minimizers for a two-dimensional model without surface energy ($\kappa = 0$) and exchange energy ($\alpha = 0$) when the deformation is constrained on the boundary to be affine, $y(x) = Fx$, for $F \in \mathbb{R}^{2\times 2}_+$ in the lamination convex hull of a $\mathcal{M}$ with two wells ($N = 2$) was given in [10] using the method of convex integration.

3. Compactness and closure in the set of admissible functions

In order to demonstrate the existence of minimizers, we will use the direct method of the calculus of variations. For this purpose, we will show that weak limits of elements of $\mathcal{A}$ belong to $\mathcal{A}$. To be precise, we say that the sequence $\{(y_n, m_n)\} \subset \mathcal{A}$ *converges weakly* to $(y, m) \in \mathcal{A}$ if and only if

$$y_n \rightharpoonup y \quad \text{in } W^{2,2}(\Omega; \mathbb{R}^3),$$

and

$$\begin{aligned} \chi_{\mathcal{O}(y_n)} m_n &\to \chi_{\mathcal{O}(y)} m && \text{in } L^2(\mathbb{R}^3; \mathbb{R}^3), \\ \chi_{\mathcal{O}(y_n)} \nabla_z m_n &\rightharpoonup \chi_{\mathcal{O}(y)} \nabla_z m && \text{in } L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3}). \end{aligned}$$

It is possible to explain the weak convergence defined above within the general framework of Cartesian currents (see [15]). However, since y_n, m_n enjoy so much smoothness, we shall be content with easier, more direct methods.

It is convenient to introduce another definition. Namely, we say that a sequence $\{(y_n, m_n)\} \subset \mathcal{A}$ is $\mathcal{A}$ - *bounded* if there exists a positive constant K, independent of n, such that

$$\int_\Omega \left\{|D^2 y_n(x)|^2 + |\nabla y_n(x)|^2 + \psi(\det \nabla y_n(x))\right\} \, dx + \int_{\mathcal{O}(y_n)} |\nabla_z m_n(z)|^2 \, dz \leq K. \tag{3.1}$$

We now state the main technical result of this section.

Theorem 3.1. *If a sequence $\{(y_n, m_n)\} \subset \mathcal{A}$ is $\mathcal{A}$-bounded, then there exists a subsequence (not relabeled) and $(y, m) \in \mathcal{A}$ such that (y_n, m_n) converges weakly to (y, m).*

This result is fundamental for our considerations. It guarantees the existence of candidates for minimizers, which are weak limits of minimizing sequences. We shall divide the proof into a number of tasks. We shall first deal with $\{y_n\}$ prior to discussing $\{m_n\}$.

Our main technical tool will be the distribution function. We define

$$\begin{aligned} A_t^n &= \{x \in \Omega : \det \nabla y_n(x) < t\}, \qquad t < 1, \\ B_t^n &= \{x \in \Omega : \det \nabla y_n(x) > t\}, \qquad t > 1. \end{aligned}$$

Lemma 3.1. *If the sequence $\{(y_n, m_n)\} \subset \mathcal{A}$ is $\mathcal{A}$-bounded, then*

$$|A_t^n| \leq t^q c_L^{-1} K \quad \textit{and} \quad |B_t^n| \leq t^{-q} c_L^{-1} K.$$

Proof. Our starting point is simply

$$\begin{aligned} |A_t^n| &= \textstyle\int_{A_t^n} 1 \, dx = \int_{A_t^n} \frac{t}{t} \, dx \\ &\leq t \textstyle\int_{A_t^n} \frac{1}{\det \nabla y_n} \, dx \leq t \left(\int_{A_t^n} \frac{1}{(\det \nabla y_n)^q} \, dx\right)^{\frac{1}{q}} \cdot |A_t^n|^{1-\frac{1}{q}} . \end{aligned}$$

Hence, we have by (2.7) that

$$|A_t^n| \leq t^q \int_{A_t^n} (\det \nabla y_n)^{-q} \, dx \leq t^q c_L^{-1} \int_{A_t^n} \psi(\det \nabla y_n) \, dx \leq t^q c_L^{-1} K$$

for $t < 1$.

The argument leading to the second estimate is similarly given by

$$\begin{aligned} |B_t^n| &= \textstyle\int_{B_t^n} 1 \, dx = \int_{B_t^n} \frac{t}{t} \, dx \\ &\leq t^{-1} \textstyle\int_{B_t^n} \det \nabla y_n \, dx \leq t^{-1} \left(\int_{B_t^n} (\det \nabla y_n)^q \, dx\right)^{\frac{1}{q}} \cdot |B_t^n|^{1-\frac{1}{q}} . \end{aligned}$$

Hence, we have by (2.7) that

$$|B_t^n| \leq t^{-q} \int_{B_t^n} (\det \nabla y_n)^q \, dx \leq t^{-q} c_L^{-1} \int_{B_t^n} \psi(\det \nabla y_n) \, dx \leq t^{-q} c_L^{-1} K$$

for $t > 1$. $\square$

We next state and prove a main lemma on the convergence of $\{y_n\}$. We note that the convergence result for $\det \nabla y_n$ below is better than that implied by the Sobolev embedding and compactness theorem [1, 13, 16]. It is due to the growth condition (2.7).

Lemma 3.2. *If the sequence $\{(y_n, m_n)\} \subset \mathcal{A}$ is $\mathcal{A}$-bounded, then there exists a subsequence (not relabeled) such that*

$$y_n \rightharpoonup y \quad in \quad W^{2,2}(\Omega;\mathbb{R}^3),$$

and

$$\det \nabla y_n \to \det \nabla y \quad in \quad L^p(\Omega) \quad for\ p < q, \tag{3.2}$$

$$\det \nabla y_n \rightharpoonup \det \nabla y \quad in \quad L^q(\Omega), \tag{3.3}$$

$$\int_\Omega \psi(\det \nabla y)\, dx < \infty, \tag{3.4}$$

$$\det \nabla y > 0 \quad a.e. \tag{3.5}$$

Proof. In order to show existence of a subsequence (not relabeled) $\{y_n\}_{n=1}^\infty$, which is weakly convergent in $W^{2,2}(\Omega;\mathbb{R}^3)$, it is sufficient to prove a uniform in n bound on $\|y_n\|_{W^{2,2}}$. Due to the $\mathcal{A}$-boundedness of $\{(y_n, m_n)\}$, it is enough to prove a uniform bound on $\|y_n\|_{L^2}$. This easily follows from the boundary condition (2.1), the Poincaré inequality [1,13,16], and the $\mathcal{A}$-boundedness, because we have

$$\begin{aligned}
\textstyle\int_\Omega |y_n(x)|^2\, dx \quad &\le 2\int_\Omega |y_n(x) - y_0(x)|^2\, dx + 2\int_\Omega |y_0(x)|^2\, dx \\
&\le 2C(\Omega)\int_\Omega |\nabla\left(y_n(x) - y_0(x)\right)|^2\, dx + 2\int_\Omega |y_0(x)|^2\, dx \\
&\le C\left(\int_\Omega |\nabla y_n(x)|^2\, dx + 1\right) \le C(K+1).
\end{aligned}$$

Since the sequence $\{y_n\}$ converges weakly in $W^{2,2}(\Omega;\mathbb{R}^3)$ it follows from the Sobolev embedding theorem [1] that there exists a subsequence such that $y_n \longrightarrow y$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ for $p < 6$ and also in $C^{0,\alpha}(\overline{\Omega};\mathbb{R}^3)$ for $0 < \alpha < 1 - \frac{3}{6} = \frac{1}{2}$. Thus, for another subsequence

$$\begin{aligned}
\nabla y_n &\longrightarrow \nabla y && \text{a. e.,} \\
\det \nabla y_n &\longrightarrow \det \nabla y && \text{a. e.,} \\
\det \nabla y_n &\longrightarrow \det \nabla y && \text{in } L^p(\Omega) \quad \text{for } p < 2.
\end{aligned} \tag{3.6}$$

We now show that we can improve the convergence in (3.6). For that purpose, we are going to show that $\{(\det \nabla y_n)^p\}$ is equi-integrable for any $p < q$, that is, for any given $\epsilon > 0$ there is a $\delta > 0$ such that, if $V \subset \Omega$ satisfies $|V| < \delta$, then

$$\int_V (\det \nabla y_n)^p\, dx < \epsilon.$$

Indeed, for any $V \subset \Omega$ and $t > 1$ we have that

$$\int_V (\det \nabla y_n)^p\, dx = \left(\int_{V\setminus B_t^n} + \int_{V\cap B_t^n}\right)(\det \nabla y_n)^p\, dx. \tag{3.7}$$

Due to the definition of B_t^n, we can see that

$$\begin{aligned}
\int_V (\det\nabla y_n)^p\, dx &\le t^p\, |V \setminus B_t^n| + \left(\int_{V\cap B_t^n} (\det\nabla y_n)^q\, dx\right)^{\frac{p}{q}} \cdot |B_t^n|^{1-\frac{p}{q}} \\
&\le t^p|V| + c_L^{-p/q}\left(\int_{B_t^n} \psi\left(\det \nabla y_n\right)\, dx\right)^{\frac{p}{q}} \cdot \left(t^{-q} c_L^{-1} K\right)^{\left(1-\frac{p}{q}\right)}
\end{aligned}$$

$$\leq t^p|V| + c_L^{-1}Kt^{-q+p}.$$

We can now take $t > 1$ so large that the second term is less than $\frac{\varepsilon}{2}$. Then we choose δ so small that $t^p\delta < \frac{\varepsilon}{2}$, and the claim follows since $|V| < \delta$.

After we choose an a.e.- convergent subsequence, we deduce by Vitali's Theorem that

$$(\det\nabla y_n)^p \to (\det\nabla y)^p \quad \text{in } L^1(\Omega) \quad \text{for } p < q.$$

Furthermore, we can infer that for this subsequence [3, Theorem 1]

$$\det\nabla y_n \longrightarrow \det\nabla y \;\text{ in }\; L^p(\Omega) \quad \text{for } p < q.$$

Thus, (3.2) follows.

In order to deduce (3.3), we notice that due to (2.7) the sequence $\det\nabla y_n$ is bounded in $L^q(\Omega)$. Hence, it contains a subsequence converging weakly to g. By uniqueness of the limit it follows that $g = \det\nabla y$.

To prove (3.4), we observe that since $\det\nabla y_n \to \det\nabla y$ in $L^1(\Omega)$, we have that there exists a subsequence (not relabeled) such that $\det\nabla y_n \to \det\nabla y$ a.e. Since ψ is continuous, we have also that $\psi(\det\nabla y_n) \to \psi(\det\nabla y)$ a.e. Thus, since sequence $\{(y_n, m_n)\}$ is $\mathcal{A}$-bounded, we have by Fatou's lemma [12] that

$$\int_\Omega \psi(\det\nabla y)\,dx \leq \liminf_{n\to\infty} \int_\Omega \psi(\det\nabla y_n)\,dx \leq K < \infty.$$

Our next task is to show that (3.5) holds. For this purpose, we use again the technique of distribution functions. Let us define for $t < 1$ the set

$$A_t = \{x \in \Omega :\; \det\nabla y < t\}.$$

It is clear that (3.5) holds once we establish the estimate

$$|A_t| \leq ct^q, \tag{3.8}$$

where c is a positive constant. Since

$$\det\nabla y = \det\nabla y_n + (\det\nabla y - \det\nabla y_n)$$

and $\det\nabla y_n > 0$ a.e., we have for all $n \in \mathbb{N}$ that

$$A_t \subset A_{2t}^n \cup \{x \in \Omega :\; |\det\nabla y(x) - \det\nabla y_n(x)| \geq t\} \equiv A_{2t}^n \cup E^n.$$

By Egorov's theorem, for any $t \in (0,1)$ there is a $V \subset \Omega$ such that $|V| < t^q$ and $\det\nabla y_n$ converges uniformly to $\det\nabla y$ on $\Omega \setminus V$. Hence,

$$\begin{aligned} |A_t| &\leq |A_{2t}^n| + |E^n \cap V| + |E^n \setminus V| \\ &\leq c_L^{-1}K2^qt^q + t^q + |E^n \setminus V|. \end{aligned}$$

However, for fixed t and sufficiently large n, the set $E^n \setminus V$ is empty. Thus, (3.8) holds with $c = 1 + c_L^{-1}K2^q$. Hence, (3.5) follows. □

We remark that the methods we have presented and the assumptions on ψ allow us to prove similar convergence statements for $(\det\nabla y_n)^{-1}$. Indeed, we have

Lemma 3.3. *If $1 \leq p < q$, then $(\det\nabla y_n)^{-1}$ converges to $(\det\nabla y)^{-1}$ in $L^p(\Omega)$.*

Proof. By previous results, $\det\nabla y_n$ converges a.e., and the limit $\det\nabla y$ is positive a.e. Hence, $(\det\nabla y_n)^{-1} \to (\det\nabla y)^{-1}$ a.e. Due to Vitali's convergence theorem, it is sufficient to check that the sequence $(\det\nabla y_n)^{-p}$ is equi-integrable. To prove this, we suppose that $\epsilon > 0$ is given. Then for $V \subset \Omega$ with $|V| \le \delta$, similar to argument following (3.7), we have that

$$\begin{aligned}\int_V (\det\nabla y_n)^{-p}\,dx =& \Big(\int_{V\setminus A_t^n} + \int_{V\cap A_t^n}\Big)(\det\nabla y_n)^{-p}\,dx\\ \le& t^{-p}|V| + \Big(\int_{V\cap A_t^n}(\det\nabla y_n)^{-q}\,dx\Big)^{\frac{p}{q}}|A_t^n|^{1-\frac{p}{q}}\\ \le& t^{-p}|V| + c_L^{-1}Kt^{q-p} \le \epsilon\end{aligned}$$

for a suitable choice of t and δ. □

Finally, we want to make sure that the limit mapping y is indeed one-to-one a.e.

Lemma 3.4. *Let us suppose that $\Omega \subset \mathbb{R}^3$ is open, the deformations $y_n : \Omega \to \mathbb{R}^3$ are one-to-one a.e., and $\det\nabla y_n > 0$ a.e. We also assume that the sequence y_n converges weakly in $W^{2,2}(\Omega)$ to y, $\det\nabla y_n \to \theta$ in $L^1(\Omega)$, and $\theta > 0$ a.e. Then y is one-to-one a.e. (and obviously $\det\nabla y = \theta$).*

Proof. We need a characterization of invertibility a.e. which is easy to apply to the limit y. Let us recall for that purpose the notion of Banach's indicatrix

$$N(y,\Omega,z) = \#\{x \in \Omega :\ y(x) = z\},$$

where we restrict our attention to the continuous representative of y. Of course, we have that $y(\Omega) = \{z \in \mathbb{R}^3 :\ N(y,\Omega,z) \ge 1\}$.

We claim that y is one-to-one a.e. if and only if

$$|\{z \in \mathbb{R}^3 :\ N(y,\Omega,z) \ge 2\}| = 0.$$

Indeed, let us suppose that $y|_G$ is one-to-one and G is of full measure. Since y has the Lusin property (see [14], §5.2), we deduce that $E = y(\Omega\setminus G)$ has measure zero. Moreover, $N(y,\Omega,z) \ge 2$ if and only if $z \in E$.

On the other hand, let us suppose that $E = \{z \in \mathbb{R}^3 :\ N(y,\Omega,z) \ge 2\}$ has measure zero. We set $G = \Omega \setminus y^{-1}(E)$. We must show that $|y^{-1}(E)| = 0$. By the area formula (see [14], Theorem 5.11), we can see that

$$\int_{y^{-1}(E)} \det\nabla y(x)\,dx = \int_E N(y,\Omega,z)\,dz = 0.$$

Since $\det\nabla y > 0$ a.e., we deduce that $|y^{-1}(E)| = 0$.

We shall show that

$$N(y,\Omega,z) \le 1 \quad a.e.$$

Let us take $\phi \in C_0(\mathbb{R}^3)$ such that $\phi \ge 0$. Obviously, we obtain since y_n is one-to-one a.e. that

$$\int_\Omega \phi(y_n(x))\det\nabla y_n(x)\,dx = \int_{y_n(\Omega)}\phi(z)\,dz \le \int_{\mathbb{R}^3}\phi(z)\,dz.$$

Hence,

$$\int_\Omega \phi(y(x))\det\nabla y(x)\,dx \le \int_{\mathbb{R}^3}\phi(z)\,dz.$$

Furthermore, this inequality implies that $N(y,\Omega,z) \le 1$ a.e. That is, y is one-to-one a.e. as desired. □

Our next task is to prove the convergence of the sequence of magnetization vectors $\{m_n\}$.

Lemma 3.5. *If $\{y_n, m_n\} \subset \mathcal{A}$ is a $\mathcal{A}$–bounded sequence, then there exists $m \in W^{1,2}(\mathcal{O}(y);\mathbb{R}^3)$ such that a subsequence (not relabeled) converges weakly to (y, m).*

Proof. We shall apply again the technique of distribution functions. We define the set

$$D_t^n = \{x \in \Omega : |\nabla_z m_n \circ y_n(x)| > t\}.$$

We claim that

$$|D_t^n| \leq K c_L^{-\frac{1}{q+1}} t^{-2\frac{q}{q+1}}. \tag{3.9}$$

Indeed, by the Schwartz inequality

$$\begin{aligned}
|D_t^n| &= \int_{D_t^n} 1\,dx = \int_{D_t^n} \left(\frac{|\nabla m_n| \det^{\frac{1}{2}} \nabla y_n}{|\nabla m_n| \det^{\frac{1}{2}} \nabla y_n} \right) dx \\
&\leq \left(\int_{D_t^n} |\nabla m_n|^2 \det \nabla y_n \,dx \right)^{1/2} \left(\int_{D_t^n} |\nabla m_n|^{-2} (\det \nabla y_n)^{-1} \,dx \right)^{1/2} \\
&\leq \frac{1}{t} K^{1/2} \left(\int_{D_t^n} \frac{1}{\det \nabla y_n} \,dx \right)^{1/2} \\
&\leq \frac{1}{t} K^{1/2} \left(\int_{D_t^n} \frac{1}{(\det \nabla y_n)^q} \,dx \right)^{\frac{1}{2q}} |D_t^n|^{\frac{1}{2}(1-\frac{1}{q})} \\
&\leq \frac{1}{t} K^{1/2} \left(\int_{D_t^n} c_L^{-1} \psi\left(\det \nabla y_n\right) \,dx \right)^{\frac{1}{2q}} |D_t^n|^{\frac{1}{2}(1-\frac{1}{q})} \\
&\leq \frac{1}{t} c_L^{-\frac{1}{2q}} K^{\frac{1}{2}+\frac{1}{2q}} |D_t^n|^{\frac{1}{2}(1-\frac{1}{q})}.
\end{aligned}$$

Hence, (3.9) follows.

We recall that $\mathcal{O}(y) = y(\bar{\Omega}) \setminus y(\partial\bar{\Omega})$ is open. We define the two families of sets

$$\mathcal{O}_\varepsilon(y) = \{z \in \mathcal{O}(y) : \operatorname{dist}(z, \partial\mathcal{O}(y)) > \varepsilon\} \qquad \text{and} \qquad \mathcal{O}^\epsilon(y) = \{z \in \mathbb{R}^3 : \ \operatorname{dist}(z, \mathcal{O}(y)) \leq \epsilon\}.$$

We note that

$$\bigcup_{\varepsilon > 0} \mathcal{O}_\varepsilon(y) = \mathcal{O}(y).$$

For $\varepsilon > 0$ fixed, we have that $\mathcal{O}_\varepsilon(y) \subset \mathcal{O}(y_n)$ for sufficiently large n because of the uniform convergence of y_n to y. We now claim that for any $\varepsilon > 0$ there exists a subsequence (not relabeled) such that

$$m_n \rightharpoonup m \quad \text{in} \quad W^{1,2}(\mathcal{O}_\varepsilon(y);\mathbb{R}^3). \tag{3.10}$$

We have for any $\varepsilon > 0$ that

$$\int_{\mathcal{O}_\varepsilon(y)} |\nabla_z m_n(z)|^2 \,dz \leq \int_{\mathcal{O}(y_n)} |\nabla_z m_n(z)|^2 \,dz = \int_\Omega |\nabla_z m_n|^2 \det \nabla y_n \,dx \leq K.$$

Due to the constraint (2.2), we can see that

$$\begin{aligned}
\int_{\mathcal{O}(y_n)} |m_n(z)|^2 \,dz &= \int_\Omega |m_n(y_n(x))|^2 \det \nabla y_n(x) \,dx \\
&= \tau^2 \int_\Omega (\det \nabla y_n(x))^{-1} \,dx
\end{aligned}$$

$$\leq \tau^2 \left(\int_\Omega (\det \nabla y_n(x))^{-q} \, dx \right)^{\frac{1}{q}} |\Omega|^{1-\frac{1}{q}}$$
$$\leq \tau^2 c_L^{-\frac{1}{q}} \left(\int_\Omega \psi(\det \nabla y_n(x)) \, dx \right)^{\frac{1}{q}} |\Omega|^{1-\frac{1}{q}} \leq K_1.$$

This immediately implies that there exists a subsequence (not relabeled) such that

$$m_n \rightharpoonup m \quad \text{in} \quad W^{1,2}(\mathcal{O}_\varepsilon(y); \mathbb{R}^3).$$

It thus follows that there exists a further subsequence (not relabeled) such that

$$m_n \to m \quad \text{in} \quad L^2(\mathcal{O}_\varepsilon(y); \mathbb{R}^3). \tag{3.11}$$

We shall now show that

$$\chi_{\mathcal{O}(y_n)} m_n \to \chi_{\mathcal{O}(y)} m \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^3) \qquad \text{and} \qquad \chi_{\mathcal{O}(y_n)} \nabla_z m_n \rightharpoonup \chi_{\mathcal{O}(y)} \nabla_z m \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3}).$$

We first estimate $\chi_{\mathcal{O}(y_n)} m_n - \chi_{\mathcal{O}(y)} m$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ by observing that

$$\chi_{\mathcal{O}(y_n)} m_n - \chi_{\mathcal{O}(y)} m = (\chi_{\mathcal{O}(y_n)} - \chi_{\mathcal{O}_\epsilon(y)}) m_n + \chi_{\mathcal{O}_\epsilon(y)}(m_n - m) + (\chi_{\mathcal{O}_\epsilon(y)} - \chi_{\mathcal{O}(y)}) m.$$

Thus,

$$\begin{aligned} \|\chi_{\mathcal{O}(y_n)} m_n - \chi_{\mathcal{O}(y)} m\|_{L^2(\mathbb{R}^3)} &\leq \|m_n\|_{L^2(\mathcal{O}(y_n) \triangle \mathcal{O}_\epsilon(y))} + \|m_n - m\|_{L^2(\mathcal{O}_\epsilon)} + \|m\|_{L^2(\mathcal{O}(y) \setminus \mathcal{O}_\epsilon(y))} \\ &= I + II + III. \end{aligned}$$

To estimate I, we set $\Omega_n^\epsilon := y_n^{-1}(\mathcal{O}_\epsilon(y))$. We can see using (2.2) and (2.7) that for $t < 1$

$$\begin{aligned} I^2 &= \int_\Omega (1 - \chi_{\Omega_n^\epsilon})^2 |m_n|^2 \det \nabla y_n \, dx \\ &= \left(\int_{A_t^n} + \int_{\Omega \setminus A_t^n} \right) \frac{(1 - \chi_{\Omega_n^\epsilon})^2 \tau^2}{\det \nabla y_n} \, dx \\ &\leq \tau^2 \left(\int_{A_t^n} \left(\frac{1}{\det \nabla y_n} \right)^q dx \right)^{\frac{1}{q}} |A_t^n|^{1-\frac{1}{q}} + \frac{\tau^2}{t} |\Omega \setminus (\Omega_n^\epsilon \cup A_t^n)| \\ &\leq c t^{q-1} + \frac{\tau^2}{t} |\Omega \setminus (\Omega_n^\epsilon \cup A_t^n)|. \end{aligned}$$

We first choose $t < 1$ to make the first term small, that is, less than $\frac{1}{2}(\delta/3)^2$. We then show that we can select ϵ so that the second term is less than $\frac{1}{2}(\delta/3)^2$. This would imply that $I < \delta/3$, as desired. We can do so because

$$\begin{aligned} |\mathcal{O}^\epsilon(y) \setminus \mathcal{O}_\epsilon(y)| \geq |y_n(\Omega \setminus \Omega_n^\epsilon)| &\geq |y_n(\Omega \setminus (\Omega_n^\epsilon \cup A_t^n))| \\ &= \int_{\Omega \setminus (\Omega_n^\epsilon \cup A_t^n)} \det \nabla y_n \geq t |\Omega \setminus (\Omega_n^\epsilon \cup A_t^n)|. \end{aligned}$$

Our claim follows because $|\mathcal{O}^\epsilon(y) \setminus \mathcal{O}_\epsilon(y)|$ can be made arbitrarily small, for fixed t.

For fixed $\epsilon > 0$ and for sufficiently large n, we have that $II < \delta/3$ because of (3.11). Finally, for given $\delta > 0$ one can find $\epsilon > 0$ for which $III < \delta/3$ because $|\mathcal{O}(y) \setminus \mathcal{O}_\epsilon(y)|$ can be made arbitrarily small, and integration is absolutely continuous with respect to the set of integration.

The weak convergence is slightly easier. For $\varphi \in L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3})$, we consider

$$\int_{\mathbb{R}^3} (\chi_{\mathcal{O}(y_n)} \nabla_z m_n - \chi_{\mathcal{O}(y)} \nabla_z m) \cdot \varphi \, dz$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \left[(\chi_{\mathcal{O}(y_n)} - \chi_{\mathcal{O}_\epsilon(y)}) \nabla_z m_n \cdot \varphi + \chi_{\mathcal{O}_\epsilon(y)} (\nabla_z m_n - \nabla_z m) \cdot \varphi + (\chi_{\mathcal{O}_\epsilon(y)} - \chi_{\mathcal{O}(y)}) \nabla_z m \cdot \varphi \right] dz \\
&= J + JJ + JJJ.
\end{aligned}
$$

We can see that

$$|J| \le \|\nabla_z m_n\|_{L^2(\mathcal{O}(y_n))} \|\varphi\|_{L^2(\mathcal{O}_(y_n) \setminus \mathcal{O}_\epsilon(y))}.$$

We can make $|J| < \delta/3$ by taking $\epsilon > 0$ small enough. The second term can be made small due to (3.10). Moreover, it is clear that $|JJJ| < \delta/3$ for sufficiently small $\epsilon > 0$. □

Finally, we shall demonstrate the lower semicontinuity of the mixed term $\int_\Omega |\nabla m_n|^2 \det \nabla y_n \, dx$.

Lemma 3.6. *If $\{(y_m, m_n)\} \subset \mathcal{A}$ is an $\mathcal{A}$-bounded sequence, then*

$$\int_\Omega |\nabla_z m|^2 \det \nabla y \, dx \le \liminf_{n\to\infty} \int_\Omega |\nabla_z m_n|^2 \det \nabla y_n \, dx.$$

Proof. If we take any $\delta > 0$, then there exists $\varepsilon > 0$ such that

$$\int_{\mathcal{O}_\varepsilon(y)} |\nabla_z m|^2 \, dz \ge \int_{\mathcal{O}(y)} |\nabla_z m|^2 \, dz - \delta.$$

It then follows from (3.10) that

$$
\begin{aligned}
\liminf_{n\to\infty} \int_\Omega |\nabla_z m_n|^2 \det \nabla y_n \, dx &= \liminf_{n\to\infty} \int_{\mathcal{O}(y_n)} |\nabla_z m_n|^2 \, dz \ge \liminf_{n\to\infty} \int_{\mathcal{O}_\varepsilon(y)} |\nabla_z m_n|^2 \, dz \\
&\ge \int_{\mathcal{O}_\varepsilon(y)} |\nabla_z m|^2 \, dz \ge \int_{\mathcal{O}(y)} |\nabla_z m|^2 \, dz - \delta = \int_\Omega |\nabla_z m|^2 \det \nabla y \, dx - \delta.
\end{aligned}
$$

The proof follows since δ was arbitrary. □

We are now ready for the proof of Theorem 3.1. If $\{(y_n, m_n)\} \subset \mathcal{A}$ is an $\mathcal{A}$-bounded sequence (3.1) with bound K, then by Lemmas 3.1–3.6 its subsequence converges weakly to an element (y, m) in $\mathcal{A}$ and

$$\int_\Omega \left\{ |D^2 y|^2 + |\nabla y|^2 + \psi(\det \nabla y) \right\} dx + \int_{\mathcal{O}(y)} |\nabla_z m|^2 \, dz \le K.$$

4. The existence of an energy minimizer

We are going to demonstrate a lower semicontinuity property for the energy $\mathcal{E}$. We begin with a simple observation on minimizing sequences.

Lemma 4.1. *If $\{(y_n, m_n)\} \subset \mathcal{A}$ is a minimizing sequence, then it is $\mathcal{A}$ - bounded.*

Proof. We can estimate the magnetic interaction energy term, which is the only non-positive expression in $\mathcal{E}$, by

$$
\begin{aligned}
J_n &= \left| \int_{\mathcal{O}(y_n)} h(z) \cdot m_n(z) \, dz \right| = \left| \int_\Omega h(y_n(x)) \cdot m_n(y_n(x)) \det \nabla y_n(x) \, dx \right| \\
&\le C_\varepsilon \int_{\mathcal{O}(y_n)} h^2(z) \, dz + \varepsilon \int_\Omega |m_n(y_n(x))|^2 \det \nabla y_n \, dx \\
&\le C_\varepsilon \parallel h \parallel^2_{L^2(\mathbb{R}^3)} + \varepsilon \tau^2 \int_\Omega (\det \nabla y_n)^{-1} \, dx \\
&\le C_\varepsilon \parallel h \parallel^2_{L^2(\mathbb{R}^3)} + \varepsilon \tau^2 c_L^{-\frac{1}{q}} \left(\int_\Omega \psi(\det \nabla y_n) \, dx \right)^{\frac{1}{q}} |\Omega|^{\frac{q-1}{q}}.
\end{aligned}
$$

By Young's inequality, we have that

$$J_n \le C_\varepsilon \parallel h \parallel^2_{L^2(\mathbb{R}^3)} + \varepsilon \tau^2 c_L^{-\frac{1}{q}} \left(\frac{1}{q} \int_\Omega \psi(\det \nabla y_n)\, dx + \frac{q-1}{q} |\Omega| \right).$$

Hence, due to (2.8) we have that

$$\begin{aligned}
&\int_\Omega \left[\kappa |D^2 y_n|^2 + |\nabla y_n|^2 + \psi(\det \nabla y_n) + |\nabla_z m_n \circ y_n|^2 \det \nabla y_n \right] dx \\
&\qquad \le \max\left\{ 1, \frac{1}{C_L} \right\} \mathcal{E}(y_n, m_n) + J_n + |\Omega| \\
&\qquad \le K + |\Omega| + C_\varepsilon \parallel h \parallel^2_{L^2(\mathbb{R}^3)} + \varepsilon \tau^2 c_L^{-\frac{1}{q}} \left(\frac{1}{q} \int_\Omega \psi(\det \nabla y_n)\, dx + \frac{q-1}{q} |\Omega| \right).
\end{aligned}$$

Since ε can be made arbitrarily small and since we have assumed that $h \in L^2(\mathbb{R}^3)$, our claim follows. □

We are thus ready to prove the main result of this paper, the existence of a solution to (2.9).

Theorem 4.1. *Suppose that the magnetostrictive free energy $\mathcal{E}$ is given by (2.3) and that the growth assumptions* (2.7) *and* (2.8) *hold. Then the minimum free energy satisfying the saturation constraint* (2.2) *is attained.*

Proof. We consider a minimizing sequence $\{(y_n, m_n)\} \subset \mathcal{A}$. By Lemma 4.1, $\{(y_m, m_n)\}$ is an $\mathcal{A}$-bounded sequence. Hence, by Theorem 3.1 there exists $(y, m) \in \mathcal{A}$ such that a subsequence $\{(y_{n_k}, m_{n_k})\}$ converges weakly to (y, m). It is sufficient for us to show that

$$\mathcal{E}(y, m) \le \liminf_{n \to \infty} \mathcal{E}(y_m, m_n).$$

We shall treat each term in $\mathcal{E}$ separately, because after choosing a suitable subsequence we may replace lim inf with lim.

Due to the lower semicontinuity of the norm, we see for the elastic surface energy that

$$\kappa \int_\Omega |D^2 y|^2\, dx \le \liminf_{n\to\infty} \kappa \int_\Omega |D^2 y_n|^2\, dx.$$

We recall from (2.6) that the anisotropy energy density $\Phi(\nabla y, m)$ is the sum $\Phi(\nabla y, m) = W(\nabla y, m) + \psi(\det \nabla y)$. We first show that

$$\lim_{n\to\infty} \int_\Omega W(\nabla y_n, m_n)\, dx = \int_\Omega W(\nabla y, m)\, dx. \tag{4.1}$$

Indeed, we know that $\nabla y_n \to \nabla y$ in $L^p(\Omega : \mathbb{R}^{3\times 3})$ for $p < 6$ and $m_n \circ y_n \to m \circ y$ a.e. Moreover, due to Lemma 3.3 we have that

$$|m_n \circ y_n| = \frac{\tau}{\det \nabla y_n} \to \frac{\tau}{\det \nabla y} = |m \circ y| \quad \text{ in } L^1(\Omega).$$

Thus, by [3, Theorem 1], (there is no need to extract another subsequence) we conclude that $m_n \circ y_n \to m \circ y$ in $L^1(\Omega; \mathbb{R}^3)$. Thus, by the continuity of the Nemytskii operator [24]

$$L^r(\Omega) \times L^1(\Omega) \ni (\nabla y, m) \to W(\nabla y, m) \in L^1(\Omega),$$

where $r < 6$ is the growth factor for $W(F, m)$ given by (2.8), we deduce (4.1). We finally recall that we have proved that

$$\int_\Omega \psi(\det \nabla y)\, dx \le \liminf_{n\to\infty} \int_\Omega \psi(\det \nabla y_n)\, dx.$$

For the magnetic exchange energy, we have from Lemma 3.6 that

$$\alpha \int_\Omega |\nabla m|^2 \det \nabla y \, dx \leq \liminf_{n\to\infty} \alpha \int_\Omega |\nabla m_n|^2 \det \nabla y_n \, dx.$$

We now turn to the magnetic interaction energy and observe that

$$\lim_{n\to\infty} \int_\Omega (h \circ y_n \cdot m_n \circ y_n) \det \nabla y_n \, dx = \int_\Omega (h \circ y \cdot m \circ y) \det \nabla y \, dx.$$

We recall that

$$\int_\Omega (h \circ y_n \cdot m_n \circ y_n) \det \nabla y_n \, dx = \int_{\mathcal{O}(y_n)} m_n(z) \cdot h(z) \, dz = \int_{\mathbb{R}^3} \chi_{y_n(\Omega)} m_n(z) \cdot h(z) \, dz.$$

Since $\chi_{y_n(\Omega)} m_n$ converges to $\chi_{y(\Omega)} m$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$, our claim follows.

We finally have to show the convergence of the magnetization energy $e_{mag}(y, m)$ given by (2.4). We note that for given y and m, the weak solution $\zeta \in \mathcal{H}(\mathbb{R}^3)$ of the magnetostatic equation (2.5) satisfies

$$\int_{\mathbb{R}^3} (-\nabla_z \zeta + \chi_{\mathcal{O}(y)} m) \nabla_z \eta \, dz = 0 \quad \text{for all } \eta \in \mathcal{H}(\mathbb{R}^3) \tag{4.2}$$

where

$$\mathcal{H}(\mathbb{R}^3) = \left\{ \zeta \in \mathcal{D}'(\mathbb{R}^3) : \ \nabla \zeta \in L^2, \ \int_{\mathbb{R}^3} \zeta(z) \, dz = 0 \right\}.$$

Since $\nabla_z \zeta$ for $\zeta \in \mathcal{H}(\mathbb{R}^3)$ is an $L^2(\mathbb{R}^3; \mathbb{R}^3)$ projection of $\chi_{\mathcal{O}(y)} m(z)$, we have that

$$\|\nabla_z \zeta\|_{L^2} \leq \|\chi_{\mathcal{O}(y)} m\|_{L^2};$$

and since $\chi_{\mathcal{O}(y_n)} m_n \to \chi_{\mathcal{O}(y)} m$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$, we have

$$\lim_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_z \zeta_n|^2 \, dz = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_z \zeta|^2 \, dz.$$

We have to check that weak limits also satisfy the pointwise magnetic saturation constraint (2.2). If $(y_n, m_n) \subset \mathcal{A}$ is a minimizing sequence, then the magnetic saturation constraint (2.2) follows since we showed L^2 convergence of $m_n \circ y_n$ and $\det \nabla y_n$. Namely, we have that

$$\begin{aligned} &\int_\Omega \left| (m_n \circ y_n) \det \nabla y_n - (m \circ y) \det \nabla y \right| dx \\ &\qquad \leq \| \, m_n - m \, \|_{L^2} \| \det \nabla y_n \, \|_{L^2} + \| \, m \, \|_{L^2} \| \det \nabla y_n - \det \nabla y \, \|_{L^2} \, . \end{aligned}$$

Finally, combining all of the above results we conclude that

$$\mathcal{E}(y, m) \leq \lim_{n\to\infty} \mathcal{E}(y_n, m_n),$$

that is, $(y, m) \in \mathcal{A}$ is the desired minimum of $\mathcal{E}$. □

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