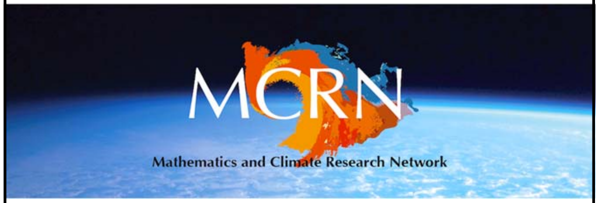
Math and Climate Seminar IMA



Joint MCRN/IMA Math and Climate Seminar
Tuesdays 11:15 – 12:05
streaming video available at
`www.ima.umn.edu`

MCRN
www.mathclimate.org

An Introduction to Energy Balance Models

Richard McGehee

Seminar on the Mathematics of Climate
IMA, MCRN, School of Mathematics
October 30, 2012

Energy Balance

Conservation of Energy

temperature change ~ energy in – energy out

short wave energy from the Sun

long wave energy from the Earth

Everything else is detail.

Energy Balance

Stefan-Boltzmann Law

$$F = \sigma T^4$$

power flux (W/m^2)

temperature (K)

Stefan-Boltzmann constant

$$\sigma \approx 5.67 \times 10^{-8}\ \mathrm{W/m^2}K^4$$

Energy Balance

Stefan-Boltzmann Law

$$F = \sigma T^4$$

power flux (W/m^2)

temperature (K)

Stefan-Boltzmann constant

$$\sigma \approx 5.67 \times 10^{-8}\ \mathrm{W/m^2}K^4$$

Example

`http://astronomybythecosmos.com/tag/sun/`

surface temperature of the Sun: 5780K

power flux: $5.67\text{x}10^{-8}$ x $(5780)^4$ = $6.33\text{x}10^7$ W/m^2

total solar power output: $6.33\text{x}10^7$ x $4\pi(r_S)^2$,
where r_S = radius of the sun = $6.96\text{x}10^8$ m

total solar output: $3.85\text{x}10^{26}$ W

200 nanoseconds = time it takes for the sun to produce the equivalent of the annual global electricity production

Energy Balance

Insolation

Solar flux at a distance r from the sun:

$$F = \frac{6.33 \times 10^7\, 4\pi r_S^2}{4\pi r^2} = 6.33 \times 10^7 \left(\frac{r_S}{r}\right)^2 \mathrm{W/m^2}$$

r_S = $6.96\text{x}10^8$ m
r = $1.5\text{x}10^{11}$ m

$$F = 1368\ \mathrm{W/m^2}$$

Power intercepted by the Earth:

$$F \times \pi r_E^2\ \mathrm{W}, \quad r_E = \text{ radius of Earth } = 6.37 \times 10^6\,\mathrm{m}$$

$$F = 1.74 \times 10^{17}\ \mathrm{W}$$

Energy Balance

Insolation

Solar flux at a distance r from the sun:

$$F = \frac{6.33\times10^7\, 4\pi r_S^2}{4\pi r^2} = 6.33\times10^7 \left(\frac{r_S}{r}\right)^2 \text{ W/m}^2$$

r_S = 6.96x10^8 m
r = 1.5x10^{11} m

$$F = 1368 \text{ W/m}^2$$

Power intercepted by the Earth:

$$F \times \pi r_E^2 \text{ W}, \quad r_E = \text{ radius of Earth } = 6.37\times10^6 \text{m}$$

$$F = 1.74\times10^{17} \text{ W}$$

Biologically Stored Energy

total coal reserves: 10^{15} kg
energy content: 3x10^7 J/kg
total energy in coal reserves: 3x10^{22} J

= ***2 days of insolation***

Energy Balance

Insolation

Global Average Insolation

intercepted flux: F = 1368 W/m^2
Earth cross-section: πr_E^2
surface area: $4\pi r_E^2$
average flux: 1368/4 = **342 W/m^2 = Q**

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

$$Q = \sigma T^4$$

$$T = \left(Q/\sigma\right)^{1/4} = \left(342/5.67\times10^{-8}\right)^{1/4}$$

$$= \boxed{279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q - \sigma T^4$$

heat capacity

stable equilibrium

Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α .

For Earth, $\alpha \approx 0.3$.

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = \left(0.7\cdot F/\sigma\right)^{1/4} = \left(0.7\cdot 342/5.67\times10^{-8}\right)^{1/4}$$

$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α .

For Earth, $\alpha \approx 0.3$.

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = \left(0.7\cdot F/\sigma\right)^{1/4} = \left(0.7\cdot 342/5.67\times10^{-8}\right)^{1/4}$$

$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Why isn't the Earth a snowball?

Energy Balance

Black Body Radiation

Planck's Function

frequency

Planck's constant

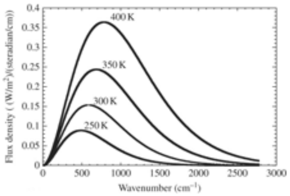
flux density ⟶

$$B(\nu,T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

temperature

speed of light

Boltzmann's constant



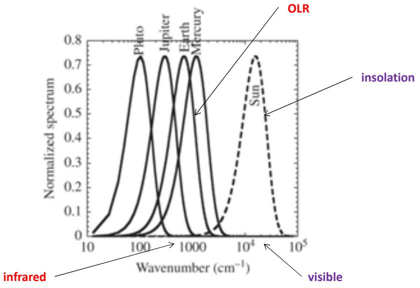
$$\text{wave number } = \frac{\nu}{c}$$

Raymond T. Pierrehumbert, *Principles of Planetary Climate*, Cambridge University Press, 2010.

Energy Balance

Insolation vs. OLR

OLR = Outgoing Longwave Radiation



OLR

insolation

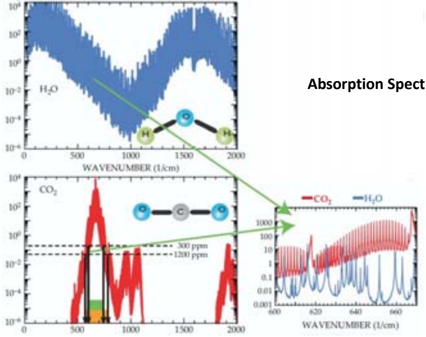
infrared

visible

Raymond T. Pierrehumbert, *Principles of Planetary Climate*, Cambridge University Press, 2010.

Energy Balance

Greenhouse Gases



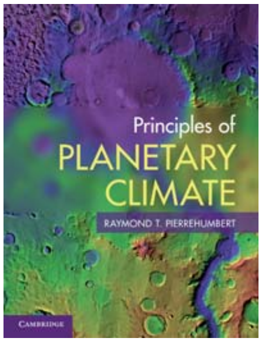
Absorption Spectra

Raymond T. Pierrehumbert, Infrared radiation and planetary temperature, *Physics Today*, 2011.

Energy Balance

Greenhouse Effect

How to learn more.



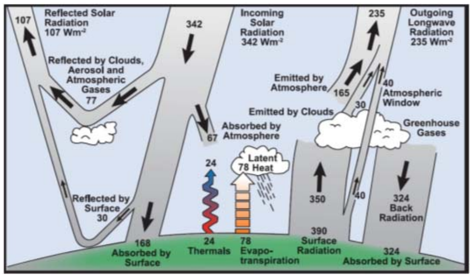
Raymond T. Pierrehumbert, Infrared radiation and planetary temperature, *Physics Today*, 2011.

Energy Balance

albedo

insolation

OLR



Historical Overview of Climate Change Science, IPCC AR4, p.96

http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

Energy Balance

OLR as a Function of Surface Temperature

$$\mathrm{OLR} = A + BT$$

A and B are determined from satellite observations.
T is surface temperature (*in Celsius*).

$$A = 202\ \mathrm{W/m^2}$$
$$B = 1.90\ \mathrm{W/m^2K}$$

Kelvin

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

Celsius

becomes

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature

Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

Equilibrium Temperature

$$T_{eq} = \frac{Q(1-\alpha) - A}{B}$$

Stable, since $B > 0$.

Ice-free planet: $\alpha = 0.32$, $T_{\mathrm{eq}} = 16$ °C
Snowball planet: $\alpha = 0.62$, $T_{\mathrm{eq}} = -38$ °C

No glacier would form on an ice-free Earth.

No glacier would melt on a snowball Earth.

Easy question:
Why do we have ice caps?

Hard question:
If Earth was ever a snowball, how did we get out?

Energy Balance

Latitude Dependence

Make T depend on $y = \sin(\mathrm{latitude})$

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation distribution

$$Q = \text{global annual average insolation} = 342\,\mathrm{W/m^2}$$

$$s(y) = \text{distribution across latitudes}\ \left(\int_0^1 s(y)dy = 1\right)$$

One can show that

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\,d\theta$$

β = obliquity = 23.5°

Chylek and Coakley’s quadratic approximation:

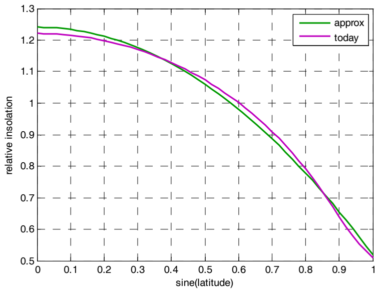
$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right)$$

Energy Balance

Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.5°



Energy Balance

Latitude Dependence

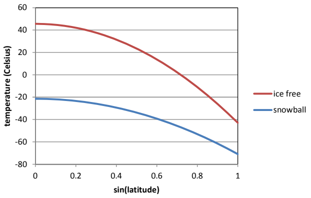
$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

Note that y is just a parameter.

Equilibrium Temperature Profile

$$T_{eq}(y) = \frac{Qs(y)(1-\alpha)-A}{B}$$

α = 0.32: ice free
α = 0.62: snowball



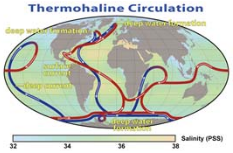
Energy Balance

What's Missing?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

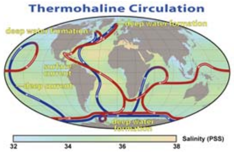
Energy Balance

What's Missing?

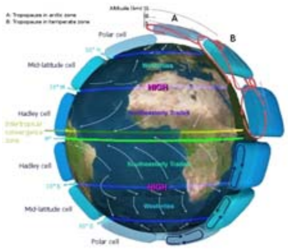


Energy Balance

What's Missing?



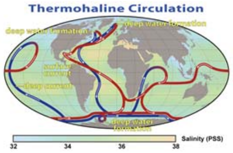
Atmospheric Circulation

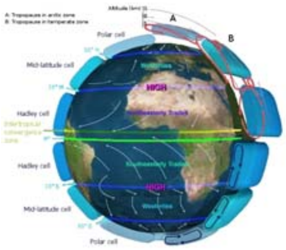


Energy Balance

What's Missing?

Weather!





Energy Balance

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + \boxed{C(\overline{T}-T)}$$

$$\overline{T}(t) = \int_0^1 T(y,t)dt$$

Weather

Second Law of Thermodynamics:
Energy travels from hot places to cold places.

Budyko's equation as a dynamical system:
T lives in a function space (temperature as a function of latitude).

Energy Balance

Budyko's Equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

albedo depends on latitude

equilibrium solution: $T = T^*(y)$

$$Qs(y)\left(1-\alpha(y)\right) - \left(A+BT^*(y)\right) + C\left(\overline{T}^* - T^*(y)\right) = 0$$

Integrate:

$$\int_0^1 \left(Qs(y)\left(1-\alpha(y)\right) - \left(A+BT^*(y)\right) + C\left(\overline{T}^* - T^*(y)\right)\right)dy = 0$$

$$Q\left(1-\overline{\alpha}\right) - \left(A+B\overline{T}^*\right) = 0$$

where $\overline{\alpha} = \int_0^1 \alpha(y)s(y)dy$

Global mean temperature at equilibrium

$$\overline{T}^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}\right) - A\right)$$

Energy Balance

Budyko's Equilibrium

$$Qs(y)\left(1-\alpha(y)\right) - \left(A+BT^*(y)\right) + C\left(\overline{T}^* - T^*(y)\right) = 0$$

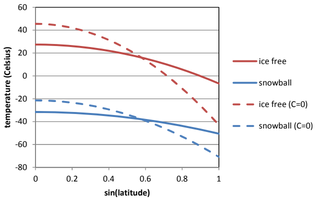
Global mean temperature at equilibrium:

$$\overline{T}^* = \frac{1}{B}\left(Q\left(1-\overline{\alpha}\right) - A\right) \qquad \left(\overline{\alpha} = \int_0^1 \alpha(y)s(y)dy\right)$$

Equilibrium temperature profile:

$$T^*(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y)\right) - A + C\overline{T}^*\right)$$

$C = 3.04$
$\alpha = 0.32$: ice free
$\alpha = 0.62$: snowball



Energy Balance

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

Next Week:

Budyko's equation as an infinite dimensional dynamical system.

Budyko's equation as a model of glacial cycles.