A Brief Introduction to Celestial Mechanics

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Outline

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Definition

Celestial mechanics is considered to be the study of the motion of objects in outer space. Historically, celestial mechanics applies the principles of physics (*classical mechanics*) to astronomical objects (e.g. stars, planets) to acquire information on these objects.

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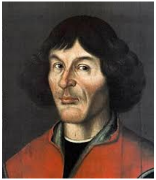
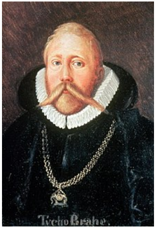
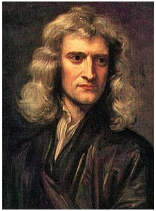
A definition we care about more:
Celestial mechanics is the study of point particles in $\mathbb{R}^3$ (mostly) moving under the influence of their mutual gravitational attraction. Generally there is an emphasis on the general orbital motions of the solar system bodies.

Timeline

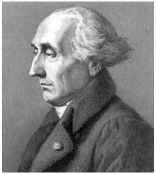
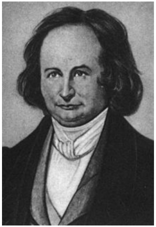
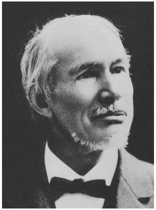
- Alexandria, ∼140CE, **Ptolemy**
- Poland, 1473 - 1543, **Nicolaus Copernicus**
- Denmark, 1546 - 1602, **Tycho Brahe**
- Germany, 1571 - 1630, **Johannes Kepler**
- England, 1643 - 1727, **Isaac Newton**

Timeline

- Switzerland, 1707 - 1783, **Leonard Euler**
- Sardinia, 1736 - 1813, **Joseph-Louis Lagrange**
- Prussia, 1804 - 1851, **Carl Jacobi**
- USA, 1838 - 1914, **George Hill**
- France, 1854 - 1912, **Henri Poincare**

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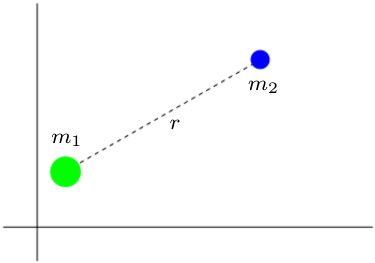
The Two-Body Problem

Let us start by looking at a simple case - **the two - body problem**. The two-body problem is to determine the position and speed of two bodies interacting with each other given their masses, initial positions, and initial velocities. The gravitational two-body problem is a special case in which the two-bodies interact by a central force F, that varies in strength as the inverse of the distance, r, between them.

- m_1, $\boldsymbol{x}_1$
- m_2, $\boldsymbol{x}_2$
- $F = \dfrac{Gm_1m_2}{r^2}$



The Two-Body Problem

The differential equations for the gravitational two-body problem are

$$\begin{cases} m_1 \ddot{\boldsymbol{x}}_1 & = F \cdot \dfrac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{r} \\ m_2 \ddot{\boldsymbol{x}}_2 & = F \cdot \dfrac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{r} \end{cases}$$

with initial conditions

$$\boldsymbol{x}_1(t_0),\ \boldsymbol{x}_2(t_0),\ \dot{\boldsymbol{x}}_1(t_0),\ \dot{\boldsymbol{x}}_2(t_0)$$

for some initial time t_0.

The Kepler Problem

We can reduce the two-body problem into a simpler problem by letting

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Now we turn our second order ODE into a system of first order ODEs and we have the familiar

$$\begin{cases} \dot{\boldsymbol{q}} &= \boldsymbol{p} \\ \dot{\boldsymbol{p}} &= -\dfrac{\mu \boldsymbol{q}}{|\boldsymbol{q}|^3} \end{cases}$$

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Conserved Quantities

A **conserved quantity** of a dynamical system is a function of the dependent variables whose value remains constant along each trajectory of the system.

Note:
Emmy Noether's first theorem states that every differentiable symmetry of the action of a physical system has a corresponding conservation law. All the conserved quantities in the Kepler problem relate to a symmetry in the system.

The Conservation of Angular Momentum

Angular momentum is the rotational analog of linear momentum. It is defined for a point particle to be the vector

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$$\frac{d}{dt}(\boldsymbol{q} \times \boldsymbol{p}) = \boldsymbol{q} \times \dot{\boldsymbol{p}} + \boldsymbol{p} \times \boldsymbol{p} = -\mu|\boldsymbol{q}|^{-3}(\boldsymbol{q} \times \boldsymbol{q}) + \boldsymbol{p} \times \boldsymbol{p} = 0.$$

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So, we have $\boldsymbol{q} \times \boldsymbol{p} = \boldsymbol{c}$, a constant vector. We will refer to $\boldsymbol{c}$ as the angular momentum.

The Conservation of Energy

The gravitational interaction as described earlier was an internal conservative force and as such we can conclude that the energy is conserved in this system. We can can find an expression for this mathematically as follows:

$$\boldsymbol{p} \cdot \dot{\boldsymbol{p}} = \boldsymbol{p} \cdot -\frac{\mu \boldsymbol{q}}{|\boldsymbol{q}|^3}$$
$$\boldsymbol{p} \cdot \dot{\boldsymbol{p}} = -\mu |\boldsymbol{q}|^{-3} (\boldsymbol{q} \cdot \dot{\boldsymbol{q}})$$
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Integrate both sides and you'll see that

$$\frac{1}{2} |\boldsymbol{p}|^2 = \frac{\mu}{|\boldsymbol{q}|} + h,$$

where h is our constant of integration and also our **energy**.

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Note: Hamiltonians!

Orbital Elements

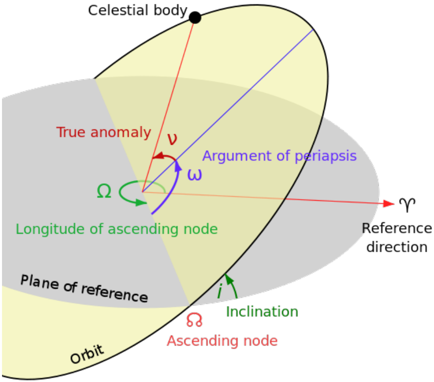

Figure: orbital plane, yellow, intersects reference plane, grey (wikipedia)

- eccentricity vector, eccentricity ($\boldsymbol{e}$)
- semi-major axis (a)
- argument of the periapsis (ω)
- true anomaly (ν)
- inclination (i)

Another Conserved Quantity!

Another constant of motion is actually the eccentric axis, $\boldsymbol{e}$. It can be show using vector identities that

$$\frac{d}{dt}\left(\frac{\boldsymbol{q}}{|\boldsymbol{q}|}\right) = \frac{\boldsymbol{c} \times \boldsymbol{q}}{|\boldsymbol{q}|^3}.$$

Then multiplying both sides of this equation by $-\mu$ we can derive the following:

$$\mu \frac{d}{dt}\left(\frac{\boldsymbol{q}}{|\boldsymbol{q}|}\right) = \dot{\boldsymbol{p}} \times \boldsymbol{c}.$$

Integrating this statement, we get

$$\mu\left(\boldsymbol{e} + \frac{\boldsymbol{q}}{|\boldsymbol{q}|}\right) = \boldsymbol{p} \times \boldsymbol{c},$$

where $\boldsymbol{e}$ is the constant of integration.

The Solution to Kepler's Problem

This can be derived by taking the dot product of both sides of our last equation with $\boldsymbol{q}$, then changing coordinates (polar coordinates), with coordinates $(|\boldsymbol{q}|, \nu)$. We find that the solution is given by

$$|\boldsymbol{q}| = \frac{|\boldsymbol{c}|^2/\mu}{1 + e\cos(\nu)}.$$

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The conic particle moves on a conic section of eccentricity e with one focus at the origin → this is Kepler's first law!

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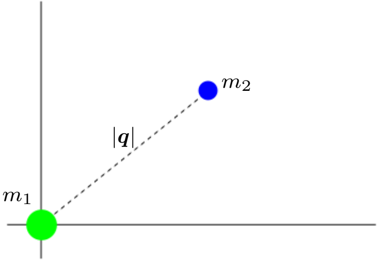
Regularization



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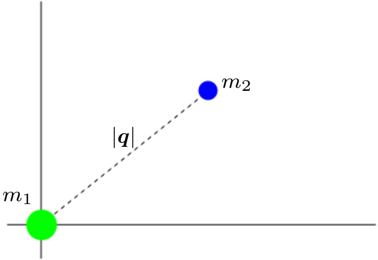
$$\Delta = \{\boldsymbol{q} \mid |\boldsymbol{q}| = 0\},$$

is the set of points where the distance to the origin is 0.

Regularization



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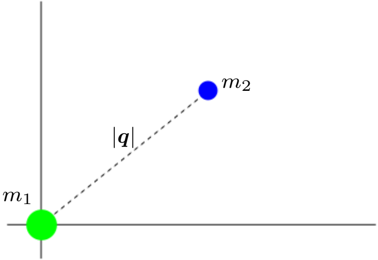

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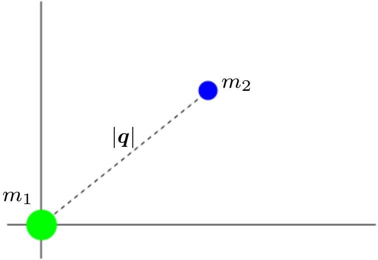

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A **regularization** is a transformation $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{u}, \boldsymbol{w})$ where $\boldsymbol{q} = 0$ corresponds to some $\boldsymbol{u} = \boldsymbol{u_0}$ and $|\boldsymbol{w}(s)| \to \boldsymbol{w_0}$ as $s \to s_0$.

Levi-Civita Regularization

Consider the problem in $\mathbb{R}^2 \cong \mathbb{C}$, that is, instead of the vector $\boldsymbol{q} = (q_1, q_2) \in \mathbb{R}^2$ we use the complex notation $q = q_1 + iq_2 \in \mathbb{C}$.

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The Levi-Civita transformation is

$$\begin{cases} q &= 2z^2 \\ p &= \dfrac{w}{\bar{z}} \end{cases}$$

where $q, p, z, w \in \mathbb{C}$. We can also utilize the use of a time change.

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New differential equations:

$$\begin{cases} z' &= \dfrac{w}{2} \\ w' &= hz \end{cases}$$

where

$$h = \frac{|p|^2}{2} - \frac{\mu}{|q|} = \frac{|w|^2}{2|z|^2} - \frac{\mu}{2|z|^2}.$$

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What is this regularization doing?

- Removes the singularity at the origin
- Singularities now lie at infinity
- Double cover
- Go around once in the z-plane $\rightarrow$ go around twice in the q-plane
- Bounce

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Other Important Topics

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- n-body problem

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- Hamiltonian systems

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- Hamiltonian systems
- Lagrangian points

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- Hamiltonian systems
- Lagrangian points
- Modern celestial mechanics

Thank You!