Permafrost Response to Climate Change via Budyko's Model*

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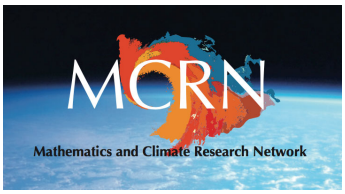
School of Mathematics
University of Minnesota

Mathematics of Climate Seminar
October 8, 2019



*Based on undergraduate projects by Aileen Zebrowski and John Nguyen.

Permafrost Response

What is permafrost?

https://www.nps.gov/gaar/learn/nature/permafrost.htm

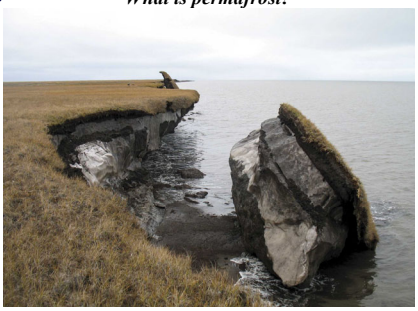
Permafrost Response

What is permafrost?

https://climatevulture.com/2016/08/28/satellite-remote-sensing-of-permafrost/

Permafrost Response

What is permafrost?



http://alaska.usgs.gov/science/interdisciplinary_science/cae/arctic_coastal_plain.php

Permafrost Response

What is permafrost?

Washington Post, Oct 4, 2019: "In fast-thawing Siberia, radical warming is warping the earth"

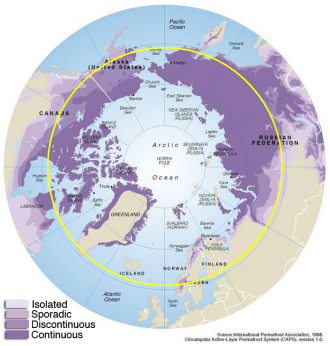
Permafrost Response

Where is the permafrost?

Average latitude of permafrost boundary: 61°

(yellow circle)

(Aileen Zebrowski)



Permafrost Response

Potential Carbon Emissions

The National Snow and Ice Data Center estimates that there are ***1400*** Gigatonnes of carbon (GtC) stored in the permafrost.

By comparison, the atmosphere currently holds about ***850*** GtC.

https://nsidc.org/cryosphere/frozenground/methane.html

Permafrost Response

Paris climate conference (COP21)

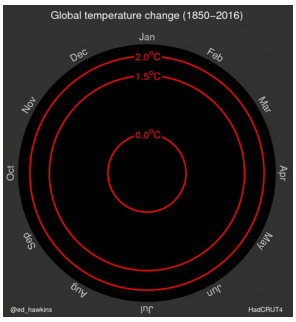
http://www.cbc.ca/radio/thecurrent/a-special-edition-of-the-current-for-november-30-2-degrees-1.3343179

Permafrost Response

Paris climate conference (COP21)

http://www.npr.org/sections/thetwo-way/2015/12/12/459502597/2-degrees-100-billion-the-world-climate-agreement-by-the-numbers

Permafrost Response



http://www.climate-lab-book.ac.uk/2016/spiralling-global-temperatures/

Permafrost Response

How much carbon would be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?

Permafrost Response

Conservation of Energy

temperature change ~ energy in – energy out

short wave energy from the Sun

long wave energy from the Earth

Everything else is detail.

Permafrost Response

Budyko's Equation

surface temperature; sin(latitude); $\overline{T} = \int_0^1 T(y)dy$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

heat capacity; insolation; albedo; OLR; heat transport

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley's quadratic approximation:

$s(y) \approx 1 + s_2\left(3y^2 - 1\right)$, where $s_2 = -0.241$

Permafrost Response

Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\overline{T}-T)$$

parameter	value	units
Q	343	Wm^{-2}
s_2	−2.41	dimensionless
A	202	Wm^{-2}
B	1.9	$\mathrm{Wm}^{-2}\mathrm{K}^{-1}$
C	3.04	$\mathrm{Wm}^{-2}\mathrm{K}^{-1}$
α_1	0.32	dimensionless
α_2	0.62	dimensionless
T_c	−10	°C

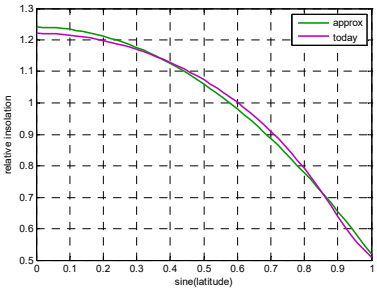
K.K. Tung, *Topics in Mathematical Modeling*, Princeton University Press, 2007, Chapter 8.

Permafrost Response

Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.4°



Permafrost Response

Equilibrium Temperature Profile

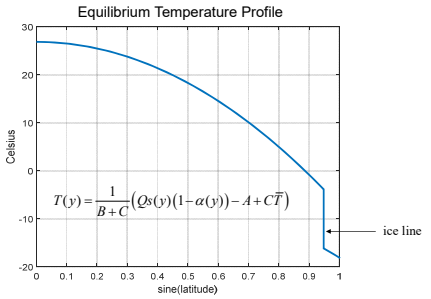
$$T(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha(y)) - A + C\overline{T}\right)$$

where $\alpha(y) = \begin{cases} \alpha_1 = 0.32, & y < \eta, \\ \alpha_2 = 0.62, & y > \eta, \end{cases}$ ← current ice boundary

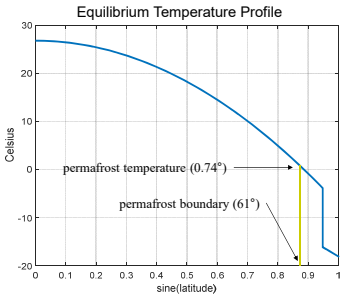
global mean temperature → $\overline{T} = \frac{1}{B}\left(Q(1-\overline{\alpha}) - A\right)$, and $\overline{\alpha} = \int_0^1 \alpha(y)s(y)dy.$ ← average albedo

$$T(y) = \begin{cases} \frac{1}{B+C}\left(Qs(y)(1-\alpha_1) - A + C\overline{T}\right), & y < \eta, \\ \frac{1}{B+C}\left(Qs(y)(1-\alpha_2) - A + C\overline{T}\right), & y > \eta. \end{cases}$$ piecewise quadratic

Permafrost Response



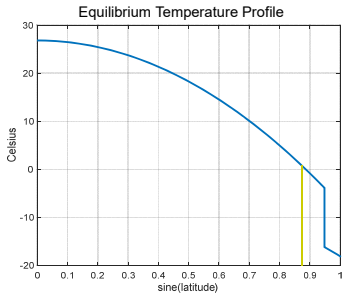
Permafrost Response



Permafrost Response

If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

First approximation: Raise the temperature profile by 2°, and use linear approximation to determine the latitude where the new temperature is the permafrost temperature.



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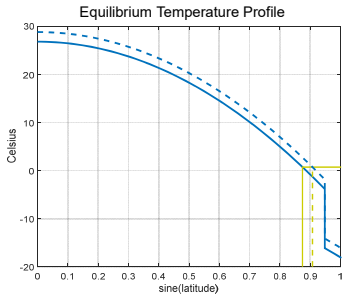

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First approximation: Raise the temperature profile by 2°, and *use linear approximation* to determine the latitude where the new temperature is the permafrost temperature.

$$T(y)=\begin{cases}\dfrac{1}{B+C}\left(Qs(y)(1-\alpha_1)-A+C\overline{T}\right), & y<\eta,\\ \dfrac{1}{B+C}\left(Qs(y)(1-\alpha_2)-A+C\overline{T}\right), & y>\eta,\end{cases}$$

$$=\begin{cases}26.85-34.13y^2, & y<0.948,\\ 1.00-19.07y^2, & y>0.948.\end{cases}$$

permafrost boundary: $y_p=\sin(61^\circ)\approx 0.875$

$$T'(y_p)=-68.26y_p=-59.70$$

$$\hat{T}(y)=T(y)+2$$

$$T(y_p)=\hat{T}(y_p+\Delta y)\approx\hat{T}(y_p)+\hat{T}'(y_p)\Delta y=T(y_p)+2+T'(y_p)\Delta y$$

$$\Delta y\approx\frac{-2}{T'(y_p)}=\frac{-2}{-59.70}=0.0335$$

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new permafrost boundary:

$$\hat{y}_p\approx 0.875+0.0335\approx 0.908$$

new permafrost boundary in degrees latitude:

$$\sin^{-1}(\hat{y}_p)\approx 65.2^\circ \text{ latitude}$$

Permafrost Response

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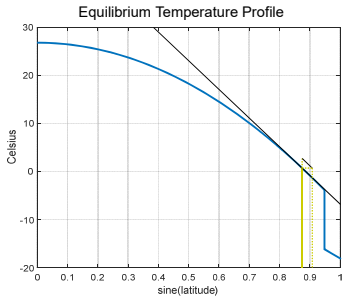

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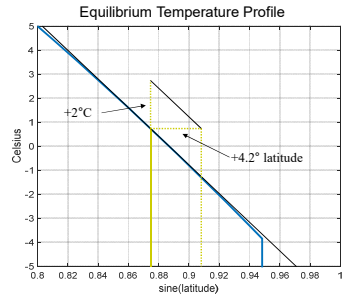
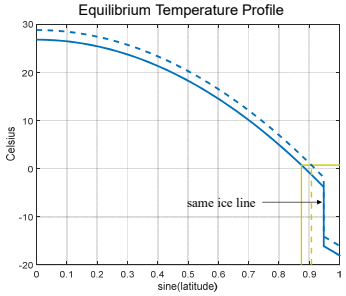


Permafrost Response

If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

We have not taken into account that the ice line might move.

Equilibrium Temperature Profile



Permafrost Response

Global Mean Temperature

$$\bar{T}(\eta)=\frac{1}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right),\ \text{where}\ \bar{\alpha}(\eta)=\int_0^1\alpha(y,\eta)s(y)dy,$$

$$\text{where}\ \alpha(y)=\begin{cases}\alpha_1=0.32, & y<\eta,\\ \alpha_2=0.62, & y>\eta,\end{cases}\quad\text{ice line}$$

The ice line is determined by the assumption that the average temperature across the ice line is T_c, usually take to be $-10°\text{C}$. This condition reduces to*

$$\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)\right)-A+C\bar{T}(\eta)\Big)=T_c,\ \text{where}\ \alpha_0=\frac{1}{2}(\alpha_1+\alpha_2)$$

outgoing long wave radiation varies with greenhouse gases.

$$h(\eta,A)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)\right)-A+\frac{C}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right)\Big)-T_c=0$$

*e.g., McGehee & Widiasih 2014, *SIAM J. Applied Dynamical Systems* **13**, pp 518-536.

Permafrost Response

How to Proceed?

1. Determine how the ice line varies with the parameter A.
2. Determine the change in A giving an increase of 2 degrees Celsius in the global mean temperature.
3. Determine the change in the location of the permafrost boundary given the change in A.

Permafrost Response

Step 1

Solve for η as a function of A:

$$h(\eta,A)=\frac{1}{B+C}\left(Qs(\eta)\left(1-\alpha_0\right)\right)-A+\frac{C}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right)\Big)-T_c=0,$$

where

$$\bar{\alpha}(\eta)=\int_0^\eta\alpha_1 s(y)dy+\int_\eta^1\alpha_2 s(y)dy$$

$$=\alpha_1\int_0^\eta s(y)dy+\alpha_2\left(1-\int_0^\eta s(y)dy\right)=\alpha_2-(\alpha_2-\alpha_1)\int_0^\eta s(y)dy$$

Numerically,

$$h(\eta,A)=h_0(\eta)-0.5236A,\ \text{where}\ h_0(\eta)=-8.0309\eta^3-26.6024\eta^2+41.3542\eta+97.8714$$

$$h_0'(\eta)\frac{d\eta}{dA}-0.5236=0$$

Evaluate at $\eta=0.9483$: $\dfrac{d\eta}{dA}=\dfrac{0.5236}{-30.7672}=-0.0171$ $\boxed{\dfrac{d\eta}{dA}=-0.0171}$

Permafrost Response

Step 2

Compute $\dfrac{d\bar{T}}{dA}$:

$$\bar{T}(\eta,A)=\frac{1}{B}\left(Q\left(1-\bar{\alpha}(\eta)\right)-A\right),\ \text{where}\ \bar{\alpha}(\eta)=\alpha_2-(\alpha_2-\alpha_1)\int_0^\eta s(y)dy$$

$$\frac{d\bar{T}}{dA}=\frac{\partial\bar{T}}{\partial\eta}\frac{d\eta}{dA}+\frac{\partial\bar{T}}{\partial A}=-\frac{Q}{B}\bar{\alpha}'(\eta)\frac{d\eta}{dA}-\frac{1}{B}=\frac{Q}{B}(\alpha_2-\alpha_1)s(\eta)\frac{d\eta}{dA}-\frac{1}{B}$$

Evaluate at $\eta=0.9483$: $\dfrac{d\bar{T}}{dA}=-1.09172$

Change in A to increase T by 2 degrees:

$$\Delta A\approx\frac{\Delta T}{-1.09172}=\frac{2}{-1.09172}=-1.832$$

$\boxed{\Delta A\approx-1.832}$

Permafrost Response

Step 3

Compute the change in y_p:

current temperature profile

$$T(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-A+C\bar{T}\right),\quad y<\eta$$

$$=26.85-34.13y^2$$

$\Delta\bar{T}=2$

$\Delta A\approx-1.832$

new temperature profile

$$\hat{T}(y)=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-(A+\Delta A)+C(\bar{T}+\Delta\bar{T})\right)$$

$$=\frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-A+C\bar{T}\right)+\frac{C\Delta\bar{T}-A}{B+C}$$

$$=T(y)+1.60$$

permafrost boundary

$y_p=\sin(61^\circ)\approx0.875$

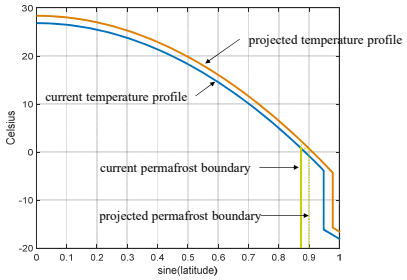
$$\Delta y\approx\frac{-1.60}{T'(y_p)}=\frac{-1.60}{-59.70}=0.027$$

as before, but with 1.6 instead of 2

new permafrost boundary $\hat{y}_p=y_p+\Delta y=0.902$, corresponding to $\boxed{64.4^\circ\ \text{latitude}}$

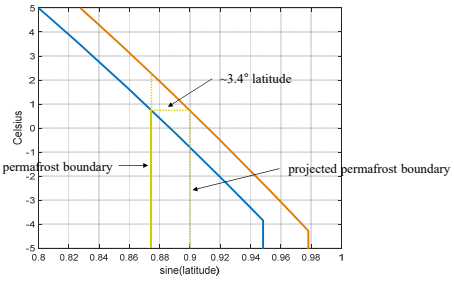
Permafrost Response

Equilibrium Temperature Profile



Permafrost Response

Equilibrium Temperature Profile

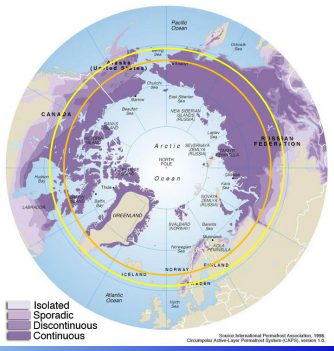


Permafrost Response

Where is the permafrost?

Average latitude of permafrost boundary: 61°
(yellow circle)

Projected permafrost boundary
(orange circle)



Permafrost Response

How much carbon would be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?

Recall that the surface area is proportional to y, the sine of the latitude.

Current permafrost boundary: $y_p = \sin(61^\circ) \approx 0.875$

Proportion of globe cover by permafrost: $1 - y_p = 0.125$

$\Delta y \approx 0.027$

Proportion of permafrost melted: $\dfrac{0.027}{0.125} = 0.216$

Amount of carbon released: $0.216 \times 1400 = \boxed{302 \text{ GtC}}$

Total fossil fuel emissions since 1751: 375 GtC

To hold the GMT at 2°C, we will have to withdraw 300 GtC from the atmosphere as the permafrost melts.

Permafrost Response

Future Projects

Budyko's model includes ice-albedo feedback, but not carbon feedback.

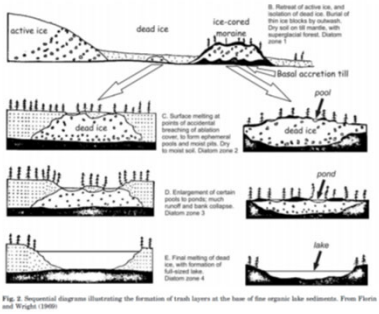
- Can we modify the model to include the effects of permafrost melt on atmospheric carbon?
- Could we use the data we have about current permafrost to model the glacial retreats during the Pleistocene?*
- To what extent was the "dead ice" in the Holocene similar to today's permafrost?**

*e.g., J.A. Walsh, E. Widiasih, J. Hahn & R. McGehee, *Nonlinearity* **29**, 1843-1864 (2016).

H. Wright & I. Stefanova, *Acta Palaeobotanica* **44, 141–146 (2004).

Permafrost Response

Future Projects



Permafrost Response

Future Projects

- Could we use the data we have about current permafrost to model the glacial retreats during the Pleistocene?
- To what extent was the "dead ice" in the Holocene similar to today's permafrost?
- Can the ^{13}C data be reconciled with the Budyko model?
- How fast will the permafrost melt?*

**Ask Kaitlin Hill*

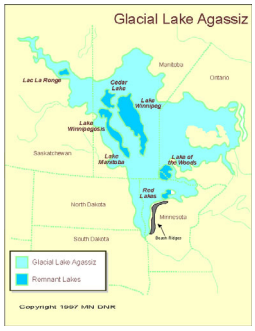
Permafrost Response

Siberian Permafrost Sinkhole

Permafrost Response

Siberian Permafrost Sinkhole

Permafrost Response



http://www.rootsweb.ancestry.com/~ndpembin/html/lake_agassiz.htm

Permafrost Response

Could this be a picture of the North shore of Lake Agassiz?