# Climate, Non-Smooth Dynamics, and Conley Theory

Cameron Thieme

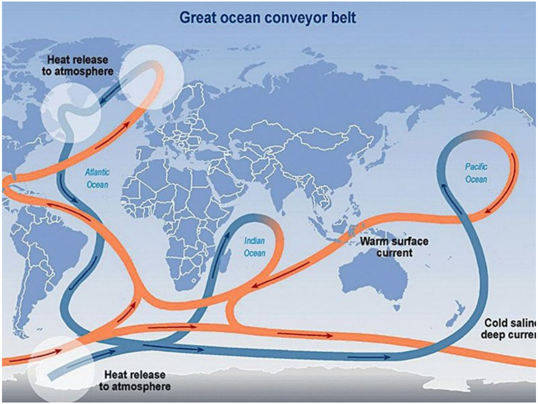
University of Minnesota

*thiem019@umn.edu*

October 15, 2019

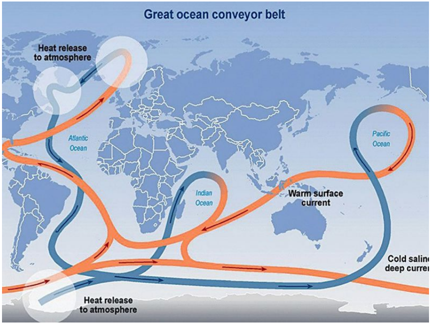
# Overview

# Atlantic Meridional Overturning Circulation




The Atlantic Meridional Overturning Circulation is a component of ocean currents that moves heat and water around the Earth. Its behavior has a strong influence on our climate.
Image: [21]

# Welander's Model: Atlantic Overturning Circulation




There is strong evidence that AMOC has changed convective strength in the past.
Welander's goal: Prove these changes could be internally driven, instead of relying on outside forcing [20].

# Welander's Model

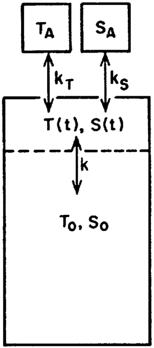



Figure: Deep Ocean and Shallow Ocean [20]

Ocean circulation box model: Planar system, salt (S) and temperature (T) are dynamic variables.

Welander's goal: Show internally driven ocean convection strength oscillations, instead of relying on outside forcing.

# Welander's Model

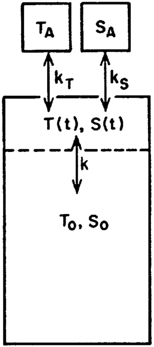



Figure: Deep Ocean and Shallow Ocean [20]

$$\dot{T} = k_T(T_A - T) - k(\rho)T$$
$$\dot{S} = k_S(S_A - S) - k(\rho)S$$
$$\rho = -\alpha T + \gamma S$$

Smooth Version:

$$k(\rho) = \frac{1}{\pi}\tan^{-1}\left(\frac{\rho - \epsilon}{a}\right) + \frac{1}{2}$$

Nonsmooth Version:

$$k(\rho) = \begin{cases} k_1 & \rho > \epsilon \\ 0, & \rho < \epsilon \end{cases}$$

$\Sigma$: Line $\rho = \epsilon$

# Welander's Model: Smooth Version

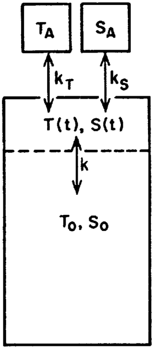



Figure: Deep Ocean and Shallow Ocean [20]

When a smooth $k$ is used, the system can be analyzed using traditional methods.

Welander uses Poincare-Bendixson to find a periodic orbit and get a proof-of-concept for his convective oscillation idea.

# Welander's Model: Non-Smooth Version

When a non-smooth $k$ is used, the system cannot be analyzed using traditional methods.

It is clear that this model was Welander's original motivation, however.

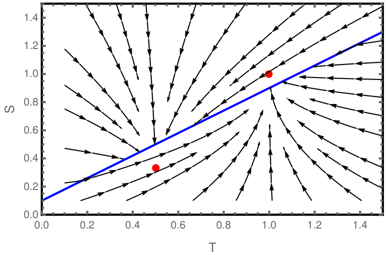



Figure: Non-Smooth Version of Welander's model. The red dots are equilibria on opposite sides of the switching boundary. [10]

# Different Dynamics for Different Regions: Filippov Systems

Welander's non-smooth model is an example of a Filippov System.

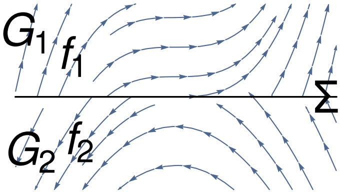



Figure: A planar Filippov system with $\mathbb{R}^2$ split into two regions.

$$\dot{x} \in F(x) = \begin{cases} f_1(x), & x \in G_1 \\ f_2(x), & x \in G_2 \\ \{\alpha f_2(x) + (1-\alpha) f_1(x) : \alpha \in [0,1]\} & x \in \Sigma \end{cases}$$

# Filippov Systems and Climate

Many other climate models are also Filippov Systems:

- Earth's surface albedo [4][6]
- Ecological Decision Making [17]
- Socio-Economic Decision Making [19]
- Layers in the Atmosphere? (Yorkinoy Shermatova)

# Periodic Orbit in the Non-Smooth Welander Model

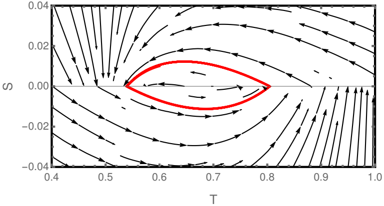



Figure: Periodic orbit in the non-smooth model.

Julie Leifeld used methods from Filippov's book to prove that the periodic orbit exists in the non-smooth model. [10]

# Periodic Orbit in the Non-Smooth Welander Model

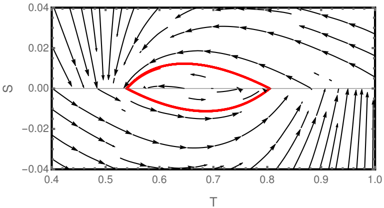



Figure: Periodic orbit in the non-smooth model.

Problems:

- The methods Dr. Leifeld used are very ad-hoc (cannot be easily applied to other models).
- There are few known techniques for analysis in very high dimensions.

# Conley Index Theory for Filippov Systems?

- Conley Index Theory is a powerful, (relatively) easy to use tool that gives robust topological information about a system.
- It works in arbitrarily high dimensions.
- It *seems* to work for Filippov systems.

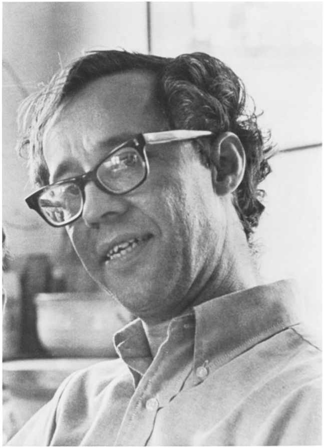


Figure: Charles Cameron Conley

# Conley Theory works with Welander

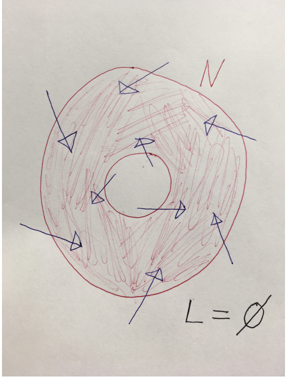



Figure: In the Welander Model, we can choose $N$ to be homeomorphic to an annulus, and it has an empty exit set $L$, so the Conley Index is of a circle and a disjoint point.

- A compact isolating neighborhood $N$ (no invariant points on boundary).
- The exit set $L \subset \partial N$
- The Conley Index is the homology sequence associated to the pair $(N/L, [L])$.

# Can Conley Theory be Extended to Filippov Systems?

- For all of our well-understood dynamics, Conley Theory seems to work, but we cannot prove (yet) that it works in general.
- Key Issue: Conley Theory requires a **flow**, but Filippov systems do not give rise to flows.
- Richard McGehee's Solution: the **multiflow**.

# Flows and Differential Equations

A **flow** is a continuous map $\varphi : \mathbb{R} \times X \to X$ satisfying the group properties

- $\varphi(0, x) = x$
- $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$

The flow relates to the differential equation

$$\dot{x} = f(x)$$

by letting $\varphi(t, x_0)$ correspond to the solution $x(t)$ with the initial condition $x(0) = x_0$.

# Why Can't Filippov Systems give Flows?

Filippov systems have:

- Intersecting trajectories
- Non-unique solutions

This prevents Filippov systems from being flows:

- No group action
- Cannot be a map

This is where multiflows come in, but let's look at the behvaior of Filippov systems first a bit.

# Behavior Near Splitting Boundary

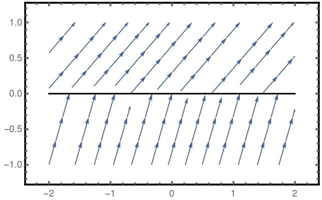



Figure: Crossing Region

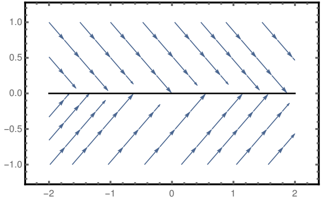



Figure: Attracting Region

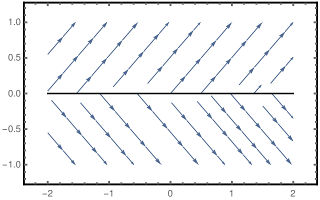



Figure: Repelling Region

# Intersecting Trajectories

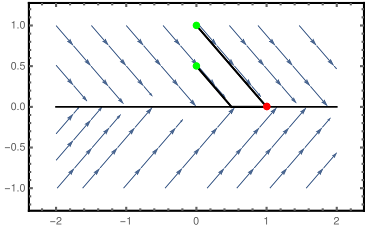



Figure: Intersecting Trajectories in a simple Filippov System

Cannot obey group properties:

$$\phi_t(\phi_{-t}(x)) = \phi_{t-t}(x) \neq \phi_0(x) = x$$

# Intersecting Trajectories

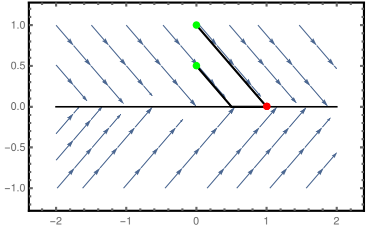



Figure: Intersecting Trajectories in a Filippov System

Solution: Monoid Action (Semiflow)

# Multiple Solutions

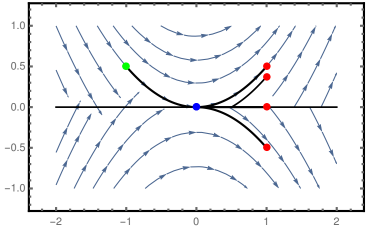



Figure: Four different solutions of a Filippov system

$$(\dot{x}, y) \in H(x,y) := \begin{cases} \{(1,x)\}, & y > 0 \\ \{(1,\beta) : \beta \in [-x,x]\} & y = 0 \\ \{(1,-x)\}, & y < 0 \end{cases}$$

# Multiflows

A multiflow is an object that is intended to generalize the concept of flows to Filippov systems.

Before we define multiflows, we need some background.

# Relations

A **relation** on a topological space $X$ is a subset of $X \times X$.

If $F$ and $G$ are both relations on $X$, then we can define the composition:

$$F \circ G = \{(x, z) \in X \times X : \exists y \in X \text{ s.t. } (x, y) \in G, (y, z) \in F\}$$

# The Closed Graph Theorem

Let $X$ be a topological space and let $Y$ be a Hausdorff space.

$f : X \to Y$ is continuous

$\downarrow$

The graph of $f$ is closed

# The Closed Graph Theorem

Let $X$ be a topological space and let $Y$ be a compact Hausdorff space.

$$f : X \to Y \text{ is continuous}$$

$$\Updownarrow$$

$$\text{The graph of } f \text{ is closed}$$

# Graph of a Flow

The graph of a **flow** $\phi$ on a compact set $X$ is a closed subset of $\mathbb{R} \times X \times X$ such that for each $t \in \mathbb{R}$, $\phi^t$ contains exactly one pair $(x, y) \in X \times X$ for each $x \in X$ and the group properties hold:

- $\phi^0 = \{(x, x) : x \in X\}$
- $\phi^{t+s} = \phi^t \circ \phi^s$

Where $\phi^t := \{(x, y) \in X \times X : (t, x, y) \in \phi\}$

# Can we modify flows to fit Filippov Systems?

~~The graph of a **flow**~~ $\phi$ on a compact set $X$ is a closed subset of $\mathbb{R}^+ \times X \times X$ such that ~~for each $t \in \mathbb{R}$, $\phi^t$ contains exactly one pair $(x, y) \in X \times X$ for each $x \in X$ and~~ the ~~group~~ *monoid* properties hold:

- $\phi^0 = \{(x, x) : x \in X\}$
- $\phi^{t+s} = \phi^t \circ \phi^s$

Where $\phi^t := \{(x, y) \in X \times X : (t, x, y) \in \phi\}$

# Multiflows

A **multiflow** $\Phi$ on a compact space $X$ is a closed subset of $\mathbb{R}^+ \times X \times X$ satisfying the monoid properties:

- $\Phi^0 = \{(x, x) : x \in X\}$
- $\Phi^{t+s} = \phi^t \circ \phi^s$

Where $\Phi^t := \{(x, y) \in X \times X : (t, x, y) \in \Phi\}$

# Filippov Systems give rise to Multiflows

**Theorem:** Let $\dot{x} \in F(x)$ be a Filippov system on an open domain $G \subset \mathbb{R}^n$, and let $K \subset G$ be compact. Let $\Phi$ be the set of all points

$$\{(T, a, b) \in \mathbb{R}^+ \times K \times K\}$$

such that there exists a solution $x : [0, T] \to K$ satisfying $x(0) = a$ and $x(T) = b$.

**Then the set $\Phi$ is a multiflow over $K$.**

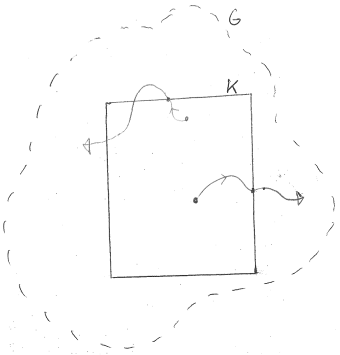



Figure: Once solutions leave $K$, they are no longer included in $\Phi$.

# Nearby Smooth Systems and Conley Theory

Conley Theory for multiflows: still in progress.

Since Conley Theory is robust under perturbation, if we can extend it we can get information about nearby smooth systems.

# Smooth Systems that Limit to Filippov Systems

Researchers are often interested in how well a non-smooth system approximates a limiting smooth system, much like in the Welander model:

Smooth Version:

$$k(\rho) = \frac{1}{\pi}\tan^{-1}\left(\frac{\rho - \epsilon}{a}\right) + \frac{1}{2}$$

Nonsmooth Version:

$$k(\rho) = \begin{cases} k_1 & \rho > \epsilon \\ 0, & \rho < \epsilon \end{cases}$$

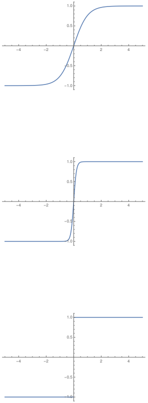


Figure: As $\alpha \to \infty$, $\tanh \alpha x$ limits to a piecewise function.

# Differential Inclusions

Filippov Systems are actually **differential inclusions**

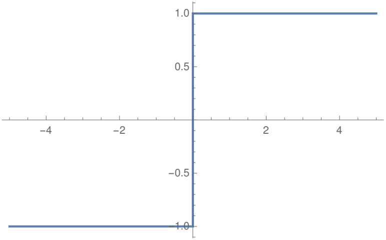
$$\dot{x} \in F(x)$$

where $F$ is a set-valued map.

A solution is an absolutely continuous function satisfying

$$\frac{d}{dt}x(t) \in F(x(t))$$

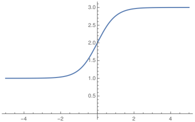
almost everywhere on some interval $I \in \mathbb{R}$.




$$F(x) = \begin{cases} -1, & x < 0 \\ [-1, 1], & x = 0 \\ 1, & x > 0 \end{cases}$$

# Jeffrey's Objection

Mike Jeffrey showed that infinitely many smooth systems can limit to the same piecewise system, but their behavior near the switching boundary can be different. [7]



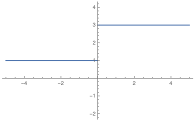




Figure: Pointwise limit of both.

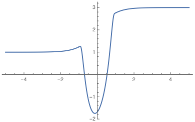



Figure: The function $tanh(\alpha x) + 2$ and the same function modified by a smooth mollifier.

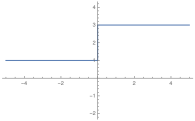
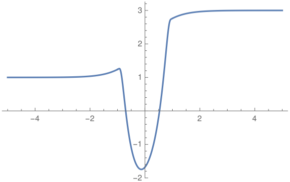
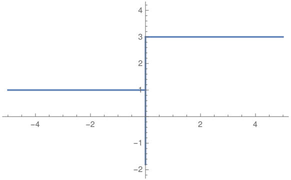



Figure: Convex combination to produce $F(x)$ from discontinuous $f$ above.

# Use Hausdorff Metric to Define "Nearby"

The convex combination method is popular, but Filippov worked in more generality. Using the Hausdorff metric we can define "nearby" more appropriately for this setting.



# References I


[1] Ball. *Continuity properties and global attractors of generalized semiflows and the Navier Stokes equations*. J. Nonlinear Sci. 7(5): 475502, 1997.

[2] Bernardo, Budd, Champneys & Kowalczyk. *Piecewise-smooth Dynamical Systems Theory and Applications*. Springer, 2008.

[3] CARABALLO, MARN-RUBIO & ROBINSON *A Comparison between Two Theories for Multi-Valued Semiflows and Their Asymptotic Behaviour*. Set-Valued Analysis 11: 297322, 2003.

[4] Engler, Kaper *Mathematics and Climate*. SIAM, 2013.

[5] Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Acad. Pub., 1988.

[6] Hartmann. *Global Physical Climatology*. Academic Press, 1994.

[7] Jeffrey. *Hidden Dynamics in Models of Discontinuity and Switching*. Physica D, 273:34-45, 2014.

[8] Jeffrey. *Ghosts of Departed Quantities in Switching and Transitions*. SIAM Review, 2018.

[9] Kuznetsov , Rinaldi & Gragnani. *One Parameter Bifurcations in Planar Filippov Systems* Int. Jnl. Bif. and Chaos, 13(08), 2003.

[10] Leifeld. *Smooth and Nonsmooth Bifurcations in Welanders Convection Model*. Thesis, University of Minnesota, 2016.

[11] Leifeld. Latex Code. Personal Communication, 2018.

[12] McGehee. Personal Communication, 2017-2018.

[13] Meyer. Personal Communication, 2017-2018.

[14] Melnik & Valero. *On Attractors of Multivalued Semi-Flows and Differential Inclusions*. Set-Valued Analysis 6: 83111, 1998.

# References II

[15] Mischaikow & Mrozek *Conley Index Theory*. 1999.

[16] Neegard. Thesis, University of Minnesota, 2018.

[17] Oyama. *Lecture Notes on Set-Valued Dynamical Systems*. Lecture Notes, University of Tokyo, https://www.u-tokyo.ac.jp, 2014.

[18] S. H. Piltz, M. A. Porter, and P. K. Maini. *Prey switching with a linear preference trade-off*. SIAM J. Appl. Math., 13(2):658682, 2014.

[19] Roxin. *On Generalized Dynamical Systems Defined by Contingent Equations*. Journal of Differential Equations, 1: 188-205, 1965.

[20] E. Santor and L. Suchanek. *Unconventional monetary policies: evolving practices, their effects and potential costs*. Bank of Canada Review, pages 115, 2013

[21] Welander. *A Simple Heat-Salt Oscillator*. Dynamics of Atmospheres and Oceans, 6(4):233-242, 1982.

[22] Woods Hole Oceanographic Institute. *Atlantic Ocean circulation at weakest point in more than 1,500 years*. phys.org, https://phys.org/news/2018-04-atlantic-ocean-circulation-weakest-years.html