Homoclinic Phenomena for Orbits Doubly Asymptotic to an Invariant Three-Sphere

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§1. Introduction. Poincaré introduced the notion of a homoclinic point in the third volume of his book on celestial mechanics [11]. The simplest nontrivial example of a homoclinic point can be described as follows. Consider a diffeomorphism of the Euclidean plane to itself with the origin fixed. Assume that the Jacobian of the mapping at the origin has two real eigenvalues, one with modulus less than unity, the other with modulus greater than unity. The stable manifold theorem applied to this fixed point gives a smooth invariant curve, called the stable manifold of the point, characterized as the set of points tending asymptotically to the origin under iterates of the map. Similarly the set of points tending asymptotically to the origin under iterates of the inverse of the map forms a smooth curve, called the unstable manifold. An intersection of the stable and unstable manifolds at a point $p \neq 0$ is called a "homoclinic point." If the intersection is transverse, the homoclinic point is called "non-degenerate."

Many authors have studied homoclinic points, including Poincaré [11], G. D. Birkhoff [2], and Smale [13]. An exposition of much of the work, including the background for this paper, can be found in Moser's book [9]. Stated roughly, near a non-degenerate homoclinic point there can be found wild recursive orbits. More specifically, there is a Cantor set on which the mapping behaves like a sequence shift on an infinite number of symbols [9]. Such behavior we loosely refer to as "stochastic behavior."

Homoclinic orbits can also be defined for a periodic orbit of a flow. In this case, an orbit is called homoclinic if it is asymptotic both forward and backward to the same periodic orbit. It is called non-degenerate if the stable and unstable manifolds intersect transversely along the homoclinic orbit. Stochastic behavior such as described above exists for this situation.

More generally, consider an invariant set I for a smooth flow on a manifold M. We shall say a point $x \in M$ is "biasymptotic to I" if $\omega(x) \subset I$, and $\alpha(x) \subset I$, but $x \notin I$, where $\omega(x)$ is the omega-limit set of x and $\alpha(x)$ is its alpha-limit set. A natural question that one can ask is whether nearby stochastic behavior must be associated with a non-degenerate biasymptotic orbit. The answer is no. Devaney has exhibited examples where I is a single point and the stable and unstable manifolds intersect transversally, yet there is no stochastic behavior [3]. In Example 4.1 below, we exhibit an invariant set I diffeomorphic to a three sphere, whose stable and unstable manifolds intersect transversally, but for which the flow has no stochastic behavior.

We define a point $x \in M$ to be "homoclinic" to I if x is biasymptotic to I and if $\alpha(x) = \omega(x)$. Easton [4] has examined the situation where I is an invariant three-sphere and x is homoclinic to I with $\omega(x)$ an invariant torus contained in I. He shows that there exists stochastic behavior of orbits near the orbit of x provided that I is foliated by invariant tori near $\omega(x)$ and the rotation numbers on these tori change.

In §5 and §6 below we examine the situation where I is an invariant three-sphere foliated by periodic solutions and x is homoclinic to I with $\omega(x)$ an invariant circle in I. We give sufficient conditions for the existence of stochastic behavior near the orbit of x.

The model flow that we study in this paper is special and is closely related to the flow of the planar three-body problem of celestial mechanics, as we discuss in the next section. We feel that an understanding of the orbit structure of Hamiltonian perturbations of the model flow is essential to understanding certain orbit structures in the three-body problem and is particularly relevant to proving the existence of capture and oscillatory orbits for this problem.

In this paper we study only perturbations of the model flow which vanish in a neighborhood of the invariant three sphere. This is a limitation on the applicability of our work.

§2. Relation to celestial mechanics. As mentioned above, Poincaré was led to the notion of a homoclinic point by his work in celestial mechanics. Our study here is also motivated by a problem in celestial mechanics, namely the existence of so-called "oscillatory" motions in the three-body problem. Roughly speaking, an orbit of the three-body problem is called "oscillatory" if the lim sup of the particle separations is infinite, but the lim inf is finite.

Sitnikov [12] first proved the existence of oscillatory orbits in a special case of the restricted three-body problem. Alexeev [1] extended the work and related it to homoclinic phenomena. Sitnikov and Alexeev showed that oscillatory orbits are related to "parabolic" orbits, *i.e.*, orbits which approach infinity with velocity asymptotic to zero. McGehee [8] observed that one can introduce a periodic orbit at infinity whose stable and unstable manifolds are the parabolic orbits. Moser [9] showed that oscillatory orbits are a consequence of the existence of an orbit homoclinic to the periodic orbit at infinity.

The entire program as outlined above is contained in Moser's book [9] and has been carried out only for Sitnikov's special case of the three-body problem.

The present paper represents work on the question of whether the same program can be carried out for the planar three-body problem. The basic difficulty comes from the higher dimensions. Sitnikov's example is one of two degrees of freedom. Thus an energy surface is three dimensional. The planar three-body problem, after removal of the integrals and symmetries, is a problem of three degrees of freedom. Thus an energy surface is five dimensional.

For Sitnikov's problem, one introduces at infinity a single periodic orbit. Its stable and unstable manifolds are two dimensional. For the planar three-body problem, one can make similar transformations. However, in this case the corresponding object introduced at infinity is an invariant three-sphere. One can think of this as follows: Suppose that the energy of the system is negative and that two of the particles are close together forming a binary system, while the third is escaping to infinity. Along a parabolic orbit, it will escape with asymptotically zero velocity. Thus, in the limit, all of the energy of the system will be in the binary system. No other parameters of the binary will be fixed. Thus the limiting system will be a two-body problem with fixed negative energy. After a Levi-Civita [7] regularization of the double collision, the flow is equivalent to the Hopf flow on a three-sphere, *i.e.* all orbits are periodic. So far, all this can be done with transformations similar to those in [8].

For Sitnikov's example, the existence of oscillatory solutions follows from the existence of a non-degenerate orbit homoclinic to the periodic orbit at infinity. We ask the same question for the planar three-body problem: Does the existence of a non-degenerate orbit biasymptotic to the three-sphere at infinity imply the existence of oscillatory solutions? "Non-degenerate" means that the stable and unstable manifolds to the three-sphere (*i.e.* the set of parabolic orbits) intersect transversely along the biasymptotic points. Unfortunately, we do now know whether the set of parabolic orbits forms a manifold. The difficulty is that the characteristic multipliers for each periodic orbit in the three-sphere are all one. However, the same problem exists for Sitnikov's example, and the difficulty can be overcome there [8]. By analogy, one can expect that the sets of parabolic orbits form smooth manifolds for the planar three-body problem.

One should then rephrase the question of the previous paragraph as follows: Suppose one could prove for the planar problem that the asymptotic sets to the three-sphere at infinity did indeed form smooth manifolds. Suppose further that one could prove that these stable and unstable manifolds intersected transversally. Could one then conclude the existence of oscillatory solutions?

To study this question, we consider a model problem. We consider a flow on a five dimensional manifold with an invariant three-sphere on which all orbits are periodic. Our model problem is simple enough so that the stable and unstable manifolds to the three-sphere can be adequately described. We then construct an example (Example 4.1 below) of such a flow, with the stable and unstable manifolds intersecting transversally, but with no oscillatory orbits. This example indicates that the answer to the question of the previous paragraph is negative. However, our example is not Hamiltonian and in fact cannot be made Hamiltonian, as we show in Section 8. The question remains unresolved.

However, that is the wrong question to ask. As mentioned in the introduction, we feel that the existence of a homoclinic point is more relevant than the existence of a biasymptotic point. A homoclinic point is asymptotic in both time directions to the same periodic orbit at infinity. However now it is not clear what should be the appropriate nondegeneracy condition. The stable and unstable manifolds of the periodic orbit will each be two dimensional submanifolds of the five dimensional energy surface. They cannot intersect transversally. Hence we cannot define a non-degenerate homoclinic point as a transversal intersection of these stable and unstable manifolds.

We were led to the conditions given below. Roughly speaking, we wanted to define a map that would take the α-limit set of a biasymptotic point to its ω-limit set. A fixed point of this map χ (defined below in 5.8) determines a homoclinic orbit. Our definition of a "transverse" homoclinic point (Definition 5.4) gives sufficient conditions to define this map. Our definition of a "non-degenerate" homoclinic point (Definition 5.9) gives sufficient conditions to prove that the existence of such a homoclinic point is insured under certain perturbations (Theorem 7.7). The homoclinic point of the perturbed system will in general be homoclinic to a different periodic orbit in the invariant three-sphere.

Next we wanted to show the existence of oscillatory orbits. We were then led to our definition of "hyperbolic" homoclinic orbit (Definition 5.9). In Theorem 6.11 below, we show that the existence of a hyperbolic homoclinic orbit implies the existence of oscillatory orbits.

We can now ask the following question: suppose one could prove for the planar three-body problem that the stable and unstable manifolds to the three-sphere at infinity exist and are smooth. Suppose further that one could show the existence of a hyperbolic homoclinic orbit. Could one then conclude the existence of oscillatory orbits? The answer to this question requires a careful analysis of the three-body flow near infinity. In our model problem we have assumed that the flow near the three-sphere is a product flow and this simplifies the analysis. However, we consider Theorem 6.11 to be a strong indication that the answer is yes.

Assuming the answer is yes, one can ask whether it would ever be possible to check in the three-body problem all of the conditions necessary for the application of this theorem.

For a start, one needs to know quite a bit about the stable and unstable manifolds to the three-sphere, *i.e.* the parabolic orbits; it may be possible to generalize the techniques of [8] to apply to this case. Next, one needs to know the existence of a homoclinic orbit. That may be quite easy, in view of the Hamiltonian structure and some work of Moser [10] (see Section 8 below). For certain values of the energy and angular momentum, two of the particles always remain close together relative to their distance to the third [5]. Thus the problem can be regarded as a small perturbation of two decoupled two-body problems. This decoupled problem is completely integrable and the techniques used in Section 8 should allow one to conclude the existence of homoclinic orbits for the three-body problem (for these values of energy and angular momentum).

Finally, there is the problem of checking the condition of hyperbolicity. This is undoubtedly the major difficulty. Since the map χ is symplectic, it is not even "generically" hyperbolic. Hence there is no reason to expect that this condition holds for the three-body problem. However, it seems likely to us that if χ were elliptic with a twist condition, then one could still prove the existence of oscillatory orbits. Since generically symplectic maps are either hyperbolic or elliptic with a twist, one could then expect that oscillatory solutions exist for the planar three-body problem. However, it is probably very difficult to check the appropriate conditions on the map χ.

§3. The model flow perturbation problems. In this section we establish notation and describe the problems which we discuss in the following sections.

3.1 Notation: Let $v = (v_1, v_2) \in \mathbb{R}^2$ and let $z = (z_1, z_2) \in \mathbb{C}^2$ and define $\|z\| = z_1\bar{z}_1 + z_2\bar{z}_2$ where the bar denotes complex conjugation. We treat $\mathbb{R}^2 \times \mathbb{C}^2$ as a real six dimensional vector space with symplectic structure determined by the two-form

$$\Omega = dv_1 dv_2 + (2i)^{-1}(d\bar{z}_1 dz_1 + d\bar{z}_2 dz_2).$$

Choose a smooth function $H^0 : \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{R}^1$ with

$$H^0\,(v, z) = G(v) + \frac{1}{2}\,\|z\|^2 - 2$$

which satisfies the following conditions:

3.2 The Hamiltonian system of differential equations

$$\begin{aligned} \dot{v}_1 &= H^0_{v_2} = G_{v_2} \\ \dot{v}_2 &= -H^0_{v_1} = -G_{v_1} \\ \dot{z}_1 &= -2iH^0_{\bar{z}_1} = -iz_1 \\ \dot{z}_2 &= -2iH^0_{\bar{z}_2} = -iz_2 \end{aligned}$$

determined by H^0 and Ω generates a smooth flow $\varphi^0 : \mathbb{R}^2 \times \mathbb{C}^2 \times \mathbb{R}^1 \to \mathbb{R}^2 \times \mathbb{C}^2$.

Since the equations for v and for z are decoupled, the flow φ^0 may be written

$$\varphi^0((v, z), t) = ({}^1\varphi(v, t), {}^2\varphi(z, t)).$$

We refer to this flow as the *model flow*.

3.3 Zero is a regular value of H^0.

Consequently $M^0 = \{(v, z) \in \mathbb{R}^2 \times \mathbb{C}^2 : H^0(v, z) = 0\}$ is a smooth five-dimensional manifold. M^0 is invariant since H^0 is constant on orbits of φ^0 and hence φ^0 is a flow on M^0.

3.4 $H^0(v, z) = v_1 v_2 + \frac{1}{2}\|z\|^2 - 2$ provided that $\max(v_1, v_2) \leq 1$.

3.5 There exists a smooth function $T : \Sigma_u \to (0, \infty)$ such that $\varphi^0 : R \to Q$ is a diffeomorphism where

$$\Sigma_u = \{(v, z) \in M^0 : v_1 = 1, \quad |v_2| < 1\}$$
$$R = \{(v, z, t) \in \Sigma_u \times R^1 : 0 \leq t \leq T(v, z)\}$$
$$Q = \varphi^0(R).$$

3.6 The map $\psi^0 : \Sigma_u \to \mathbb{R}^2 \times \mathbb{C}^2$ defined by $\psi^0(v, z) = \varphi^0((v, z), T(v, z))$ is a diffeomorphism of Σ_u onto Σ_s where

$$\Sigma_s = \{(v, z) \in M^0 : v_2 = 1, \quad |v_1| < 1\}.$$

Define $N = \{(v, z) \in M^0 : \max(|v_1|, |v_2|) \leq 1\}$ and define a projection $\pi : \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{R}^2$ by $\pi(v, z) = v$. Let $n = \pi(N)$, $q = \pi(Q)$, $\sigma_u = \pi(\Sigma_u)$, and $\sigma_s = \pi(\Sigma_s)$. Figure 3.1 below pictures these sets together with orbits of the flow ${}^1\varphi(v, t)$.

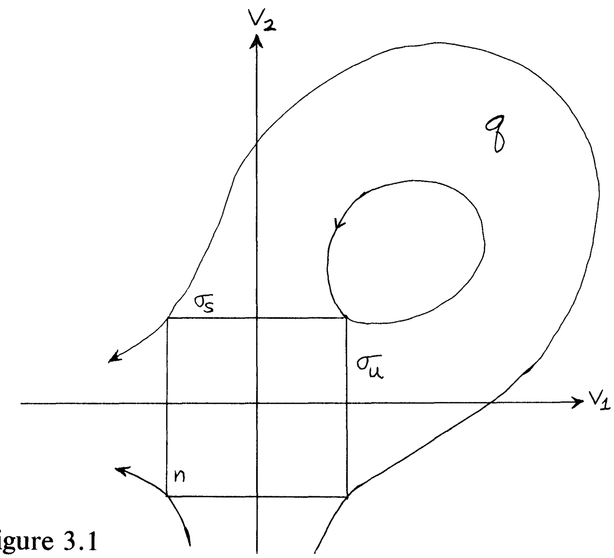

Figure 3.1

As we discussed in §2 an analysis of the three-body problem requires first techniques by which to study perturbations of the model flow φ^0 and hence of the diffeomorphism ψ^0.

3.7 *The model flow perturbation problem:* Write $x = (v, z) \in \mathbb{R}^2 \times \mathbb{C}^2$ and denote by $F^0(x)$ the vector field of (3.2). Let $f : \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{R}^2 \times \mathbb{C}^2$ be smooth. Consider the vector field $\dot{x} = F(x) = F^0(x) + f(x)$. Choose f so that conditions

(3.8) and (3.9) hold. Denote the associated flow by $\varphi : (\mathbb{R}^2 \times \mathbb{C}^2) \times \mathbb{R}^1 \to \mathbb{R}^2 \times \mathbb{C}^2$.

3.8 The support of f is a subset of Q.

3.9 There exists a smooth function $h : \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{R}^1$, such that

$$H(x) = H^0(x) + h(x)$$

is an integral for the vector field F. Furthermore, require that the support of h is a subset of Q, and zero is a regular value of H.

This last condition implies that:

3.10 $M = \{x \in \mathbb{R}^2 \times \mathbb{C}^2 : H(x) = 0\}$ is a five dimensional invariant submanifold and that φ is a flow on M. Note that M and M^0, and the corresponding vector fields, coincide except over Q. Therefore N, Σ_u and Σ_s are subsets of M, and φ and φ^0 coincide except over Q. In particular, φ and φ^0 coincide on N.

The model flow perturbation problem is the problem of studying the orbit structure of the flow φ on M.

3.11 *The model flow Hamiltonian perturbation problem:* Here we assume that the function H of 3.9 is the Hamiltonian function for the vector field F, *i.e.* $F = (H_{v_1}\ -H_{v_2},\ -2i\ H_{\bar{z}_1},\ -2i\ H_{\bar{z}_2})$. The problem again is to study the orbit structure of the flow φ.

We shall refer to 3.10 as the "perturbation problem" and to 3.11 as the "Hamiltonian perturbation problem." Henceforth, unless otherwise specified, φ will denote a flow on M as specified in 3.10.

3.12 Define $S^3 = \{(v, z) \in M : v = 0\}$. Then S^3 is an isolated invariant set for φ, and N is an isolating block for S^3. Since

$$S^3 = \{(0, z) \in \mathbb{R}^2 \times \mathbb{C}^2 : \|z\| = 2\},$$

it is a three dimensional sphere.

Following standard notation, we write

$$\omega(x) = \cap \{Cl\ (\varphi(x, [t, \infty))) : t \geq 0\},$$

$$\alpha(x) = \cap \{Cl\ (\varphi(x, (-\infty, t])) : t \leq 0\}.$$

3.13 ***Definition.*** Let $x \in M$ satisfy $\omega(x) \cap S^3 \neq \emptyset$ and $\omega(x) \cap (M - N) \neq \emptyset$. Then the orbit through x is called *ω-oscillatory*. If ω is replaced by α, then the orbit is called *α-oscillatory*. If the orbit is either α- or ω-oscillatory, it is called *oscillatory*.

3.14 ***Definition.*** $x \in M$ is called a *capture point* if $\alpha(x) \neq \emptyset$, $\alpha(x) \subset S^3$ and $\omega(x) \cap S^3 = \emptyset$. The orbit through x is called a *capture orbit*.

§4. Biasymptotic orbits. In this section we define biasymptotic orbits and we give an example of a perturbation of the model flow for which there exist nondegenerate biasymptotic orbits and for which there are no oscillatory orbits. Throughout this section φ will denote a perturbation of the model flow which satisfies the conditions 3.8 and 3.9.

4.1 ***Definition.*** $x \in M$ is called a *biasymptotic point* if $\omega(x) \subset S^3$, $\alpha(x) \subset S^3$ but $x \notin S^3$. The orbit through x is called a *biasymptotic orbit.*

4.2 ***Definitions.***

$$A_s^{\pm} = \{((v_1, v_2), z) \in N : v_1 = 0, \pm v_2 > 0\}.$$

$$A_u^{\pm} = \{((v_1, v_2), z) \in N : v_2 = 0, \pm v_1 > 0\}.$$

$$A_s = A_s^+ \cup A_s^-, \; A_u = A_u^+ \cup A_u^-.$$

$$W_s^{\pm} = \{\varphi(x, t) : x \in A_s^{\pm}, t \in \mathbb{R}^1\}.$$

$$W_u^{\pm} = \{\varphi(x, t) : x \in A_u^{\pm}, t \in \mathbb{R}^1\}.$$

$$W_s = W_s^+ \cup W_s^-, \; W_u = W_u^+ \cup W_u^-.$$

The sets W_s and W_u are called the *stable* and *unstable* manifolds of S^3 respectively and the sets A_s and A_u are called the *local stable* and *unstable manifolds* of S^3 respectively. In view of condition 3.4 the sets above can be characterized by the following equalities:

$$A_s = \{x \in N : \varphi(x, (0, \infty)) \subset N\}$$

$$A_u = \{x \in N : \varphi(x, (-\infty, 0)) \subset N\}$$

$$W_s = \{x \in M : \omega(x) \subset S^3\}$$

$$W_u = \{x \in M : \alpha(x) \subset S^3\}.$$

It follows that x is a biasymptotic point if and only if $x \in W_s \cap W_u - S^3$. Condition 3.4 implies that $W_s \cap W_u = W_s^+ \cap W_u^+$. It follows from the characterizations above that each orbit in W_s^+ intersects a_s exactly once and each orbit in W_u^+ intersects a_u exactly once,

$$a_s = A_s^+ \cap \Sigma_s \text{ and } a_u = A_u^+ \cap \Sigma_u.$$

Then each biasymptotic orbit intersects a_u in a unique point.

4.3 ***Definition.*** A *local section* (for the flow φ) is a codimension 1 submanifold of M which is everywhere transverse to the vector field F. Let Σ_1 and Σ_2 be local sections. Let D be an open subset of Σ_1, let $T: D \to R^1$ be smooth and define $\psi(x) = \varphi(x, T(x))$ for $x \in D$. Suppose that $\psi(D) \subset \Sigma_2$ and that $\psi : D \to \psi(D)$ is a diffeomorphism. Then the triple (ψ, T, D) is called a *section map from* Σ_1 *to* Σ_2. If T and D are clear from the context, then they will be omitted from the notation and ψ will be referred to as the section map.

The following proposition is standard and is a consequence of the implicit function theorem.

4.4 *Proposition.* *Suppose Σ_1 and Σ_2 are local sections and that $\varphi(x_1, t_0) = x_2$, where $x_1 \in \Sigma_1$, $x_2 \in \Sigma_2$, and $t_0 \in \mathbb{R}$. Then there exists a section map (ψ, T, D) from Σ_1 to Σ_2 with $x_1 \in D$, $T(x_1) = t_0$, and $\psi(x_1) = x_2$. Furthermore, ψ is unique up to domain.*

Remark. (ψ^0, T, Σ_u) is the unique section map from Σ_u to Σ_s, for the model flow φ^0, where (ψ^0, T, Σ_u) is defined in 3.5 and 3.6.

The following lemma is a consequence of Proposition 4.4.

4.5 *Lemma.* *Let $p \in a_u$. Then p is a biasymptotic point if and only if there exists (for φ) a section map $(\psi, T, \mathcal{D})$ from Σ_u to Σ_s such that $p \in \mathcal{D}$ and $\psi(p) \in a_s$. Furthermore, ψ is unique up to domain.*

4.6 *Definition.* Let ψ be a section map from Σ_u to Σ_s. Define

$$\Lambda_u = \psi^{-1}(a_s), \qquad \Lambda_s = \psi(a_u)$$

$$\Gamma_u = a_u \cap \Lambda_u, \qquad \Gamma_s = a_s \cap \Lambda_s.$$

Remarks. (1) Since ψ is a diffeomorphism, Λ_u and Λ_s are three dimensional submanifolds of Σ_u and Σ_s (possibly empty).

(2) Γ_u and Γ_s consist entirely of biasymptotic points and $\psi(\Gamma_u) = \Gamma_s$.

(3) Λ_u, Λ_s, Γ_u, and Γ_s all depend on ψ.

4.7 *Definition.* Let $p \in a_u$ be a biasymptotic point, let ψ be given by Lemma 4.5, and let Λ_u be defined above. We say that p is a *non-degenerate biasymptotic point* if a_u and Λ_u intersect transversally at P (as subsets of Σ_u).

Remarks. (1) Consequently, a_s and Λ_s also intersect transversally at $\psi(p)$.

(2) Γ_u and Γ_s are two dimensional submanifolds (near p and $\psi(p)$) of Σ_u and Σ_s.

(3) Non-degenerate biasymptotic points persist under perturbations. In fact, Γ_u and Γ_s persist.

(4) For the model flow all biasymptotic points are degenerate.

We are now ready to construct a perturbation of the model flow for which there exist nondegenerate biasymptotic points and for which there are no oscillatory orbits.

4.8 *Notation.* Let

$$\Sigma_u^+ = \{((v_1, v_2), z) \in \Sigma_u : v_2 > 0\}$$

$$\Sigma_s^+ = \{((v_1, v_2), z) \in \Sigma_s : v_1 > 0\}$$

$$Q = \{((v_1, v_2), z) \in M : (v_1, v_2) \in 0_q\}.$$

Conditions 3.5, 3.8, and 3.9 imply that orbits of φ enter Q through Σ_u and exit from Q through Σ_s. We require that φ be a sufficiently small perturbation of the model flow φ^0 that all orbits of φ cross Q from Σ_u to Σ_s in finite time.

The following proposition will be used to prove that the flow constructed below has no ω-oscillatory orbits.

4.9 *Proposition.* *Suppose that the orbit of φ through $x \in M$ is ω-oscillatory. Then there exist real numbers s_j, t_j for $j = 1, 2, 3, \ldots$ with $s_j < t_j < s_{j+1}$ such that for each j, $\varphi(x, [s_j, t_j]) \subset Q$ and $\varphi(x, [t_j, s_{j+1}]) \subset N$.*

Proof. We may assume without loss of generality that $x \in \Sigma_u^+$. Let $s_1 = 0$ and let t_1 denote the time it takes x to cross Q. Then $\varphi(x, t_1) \in \Sigma_s^+$. Since x is ω-oscillatory the point $\varphi(x, t_1)$ must cross N in a finite time $\mathcal{T}$. Let $s_2 = t_1 + \mathcal{T}$. By condition 3.4 we must have $\varphi(x, s_2) \in \Sigma_u^+$ or else x would not be ω-oscillatory. Continue in the same way to define the times $t_2, s_2, \cdots$. This completes the proof.

4.10 *Definitions.* Define $g : \mathbb{C}^2 \to \mathbb{R}^1$ by $g(z) = 2 - \frac{1}{2}\|z\|^2$. Define $\Sigma = \{z \in \mathbb{C}^2 : |g(z)| < 1\}$. Choose $0 < c_1 < c_2 < 1$ and choose a smooth function $\beta : [-1, 1] \to [0, 1]$ such that $\beta(t) = 1$ if $|t| \leq c_1$ and $\beta(t) = 0$ if $|t| \geq c_2$. Choose $\epsilon > 0$ and define $A : \Sigma \to \Sigma$ by $A(z_1, z_2) = (\alpha(z)z_1, \alpha^{-1}(z)z_2)$ where $\alpha(z) = 1 + \epsilon\beta(g(z))$. Define $\lambda : \Sigma \to \Sigma_u$ by $\lambda(z) = ((1, g(z)), z)$. λ is a diffeomorphism of Σ onto Σ_u. Finally, define $K : \Sigma_u \to \Sigma_u$ by $K = \lambda A \lambda^{-1}$. Let $C_j = \{((1, v_2), z) \in \Sigma_u : |v_2| < c_j\}$. Then $C_1 \subset C_2$ and

$$K(x) = \begin{Bmatrix} x \text{ if } x \in \Sigma_u - C_2 \\ ((1, g(z')), z') \text{ if } x \in C_1 \end{Bmatrix}$$

where $x = ((1, v_2), z)$ and $z^1 = ((1 + \epsilon)z_1, (1 + \epsilon)^{-1}z_2)$.

The following example can be constructed using techniques similar to those for suspending a diffeomorphism to a flow.

4.11 *Example.* Choose the vector field F in 3.7 so that conditions 3.8 and 3.9 are satisfied with $h \equiv 0$. In addition choose F so that $\psi = \psi^0 \circ K$ where φ is the flow of F and (ψ, T, Σ_u) is the section map from Σ_u to Σ_s determined by φ. The vector field F is not Hamiltonian since the section map ψ is not symplectic. We show in §8 that it is impossible to construct a Hamiltonian flow having the properties of φ.

4.12 *Proposition.* *$\psi^{-1}(a_s) \cap a_u$ is a torus. Furthermore $\psi^{-1}(a_s)$ intersects a_u transversally.*

Proof. From the definition it follows that

$$a_u \cap \psi^{-1}(a_s) = \{(v, z) \in \mathbb{R}^2 \times \mathbb{C}^2 : v = (1, 0), |z_1|^2 + |z_2|^2 = 4,$$
$$(1 + \epsilon)^2|z_1|^2 + (1 + \epsilon)^{-2}|z_2|^2 = 4\}.$$

Hence $|z_1| = 2(1 + (1 + \epsilon)^2)^{-\frac{1}{2}}$ and

$$|z_2| = 2(1 + \epsilon)(1 + (1 + \epsilon)^2)^{-\frac{1}{2}}.$$

Thus $a_u \cap \psi^{-1}(a_s)$ is by inspection a torus. The proof of transversality is omitted.

4.13 *Proposition.* *The flow φ has no oscillatory orbits.*

Proof. We show that there are no ω-oscillatory orbits. The proof for α-oscillatory is similar.

Suppose that $x \in M$ is a point on an ω-oscillatory orbit. Let s_j and t_j be given by Proposition 4.9, Let $x_j = \varphi(x, s_j) \in \Sigma_u^+$. Write

$$x_j = ((1, v_2^j), (z_1^j, z_2^j)).$$

Then

$$|z_1^{j+1}| = (1 + \epsilon\beta(v_2^j))|z_1^j|,$$
$$|z_2^{j+1}| = (1 + \epsilon\beta(v_2^j))^{-1}|z_2^j|.$$

Hence

$$|z_1^k| = |z_1^1| \prod_{j=1}^{k-1} (1 + \epsilon\beta(v_2^j)),$$

$$|z_2^k| = |z_2^1| \prod_{j=1}^{k-1} (1 + \epsilon\beta(v_2^j))^{-1}.$$

Since $2 < |z_1^j|^2 + |z_2^j|^2 < 4$,

$$\prod_{j=1}^{k-1} (1 + \epsilon\beta(v_2^j))$$

converges. Therefore there exists j^* such that $j \geq j^* \Rightarrow v_2^j > c_1$. By Proposition 4.9,

$$\varphi(x, t) \in Q \cup \{((v_1, v_2), z) \in N : v_1v_2 > c_1\},$$

for $t > t_{j^*}$. Therefore $\omega(x) \cap S^3 = \emptyset$, which is a contradiction. Hence no orbit is ω-oscillatory and the proof is complete.

§5. Homoclinic orbits. We suppose throughout this section that φ is a perturbation of the model flow which satisfies conditions 3.8 and 3.9. We define homoclinic orbits and we define local surfaces of section and section maps for the flow φ which will be used later to establish the existence of oscillatory and capture orbits.

5.1 *Definition.* $p \in M$ is called a *homoclinic point* if $\alpha(p) = \omega(p) \subset S^3$ but $p \notin S^3$. A *homoclinic orbit* is an orbit of a homoclinic point.

Since the flow φ agrees with the model flow on N it follows that the alpha and omega limit sets of a homoclinic point consist of the same periodic orbit in S^3. The flow φ on S^3 is the Hopf flow and thus S^3 consists entirely of periodic orbits of period 2π.

5.2 *Notation.* Let $S^2 = \{(w, y) \in \mathbb{C} \times \mathbb{R}^1 : |w|^2 + y^2 = 4\}$ and define $\pi : N \to S^2$ by

$$\pi(v, z) = 2\,\|z\|^{-2}(2z_1z_2, |z_1|^2 - |z_2|^2).$$

The following proposition is an immediate consequence of the definitions above.

5.3 *Proposition.*
(a) $\varphi(x, [0, t]) \subset N \Rightarrow \pi(\varphi(x, t)) = \pi(x)$.
(b) $x \in A_s \Rightarrow \pi(x) = \pi(\omega(x))$.
(c) $x \in A_u \Rightarrow \pi(x) = \pi(\alpha(x))$.
(d) $p \in a_u$ *is a homoclinic point if and only if* $\pi(p) = \pi(\psi(p))$ *where* ψ *is the section map of* φ *from* Σ_u *to* Σ_s.

5.4 *Definition.* Let $p \in a_u$ be a homoclinic point, let $q = \psi(p)$ and let $s = \pi(p)$. We say that p is a *transverse* homoclinic point if
(a) p is a non-degenerate biasymptotic point,
(b) $D\pi(p) : T_p\Gamma_u \to T_sS^2$ is an isomorphism,
(c) $D\pi(q) : T_q\Gamma_s \to T_sS^2$ is an isomorphism, where the sets Γ_u and Γ_s are defined in 4.6.

The following proposition is a consequence of the inverse function theorem.

5.6 *Proposition.* *Let* $p \in a_u$ *be a transverse homoclinic point and let* $q = \psi(p)$. *Then* $\pi : \Gamma_u \to S^2$ *and* $\pi : \Gamma_2 \to S^2$ *are local diffeomorphisms near* p *and* q *respectively.*

Restrict Γ_u and Γ_s, if necessary, so that $\pi|\Gamma_u$ and $\pi|\Gamma_s$ are diffeomorphisms and so that $\psi(\Gamma_u) = \Gamma_2$. Define

5.7 $$S_u \equiv \pi(\Gamma_u), \qquad S_s \equiv \pi(\Gamma_s).$$

Then define the diffeomorphism χ so that the following diagram commutes

5.8 $$\begin{array}{ccc} \Gamma_u & \xrightarrow{\psi} & \Gamma_s \\ \pi \downarrow & & \downarrow \pi \\ S_u & \xrightarrow{\chi} & S_s \end{array} \quad .$$

Note that, if $p \in a_u$ is a homoclinic point and if $s = \pi(p)$, then $\chi(s) = s$.

5.9 *Definition.* Let $p \in a_u$ be a transverse homoclinic point and let $s = \pi(p)$. If $D\chi(s)$ does not have eigenvalue $+1$, then we shall call p a *non-degenerate* homoclinic point. If $D\chi(s)$ is hyperbolic, then we shall call p a *hyperbolic* homoclinic point.

So far we have used the sections Σ_u and Σ_s to describe homoclinic points. It will be convenient in what follows to introduce a section Σ' which is transverse to the flow on S^3.

For the remainder of this section, $p \in a_u$ will be a fixed homoclinic point and $(\psi, T, \mathscr{D})$ will be the section map from Σ_u to Σ_s given by Lemma 4.5. Let $t_0 = T(p)$. If necessary, we may rotate the coordinate system on S^3 so that

$$\omega(p) = \alpha(p) = \{(v, (z_1, z_2)) \in M : v = 0, z_1 = 0\}.$$

5.10 *Definition.* Let $s_0 \equiv (0, -2) \in S^2$ and let $s_1 \equiv (0, +2) \in S^2$. Then $\pi(p) = \pi(\omega(p)) = \pi(\alpha(p)) = s_0$. Define the local section

$$\Sigma' = \{(v, (z_1, z_2)) \in \text{int}\,(N) : \text{Re}(z_2) > 0 \text{ and } \text{Im}(z_2) = 0\}.$$

Note that every orbit except one on S^3 hits Σ'. The exceptional orbit is

$$\{(v, (z_1, z_2)) \in M : v = 0, \quad z_2 = 0\},$$

which projects to s_1.

In what follows, the reader is asked to bear with the notation which places a prime in the upper left hand corner of every symbol. Soon a new coordinate system will be introduced on Σ' and the primes will be dropped.

Let Θ' denote the section map obtained by following the flow in N from one crossing of Σ' to the next. The domain of Θ' is

$$\mathscr{D}_{\Theta'} \equiv \{((v_1, v_2), z) \in \Sigma' : |v_1| < \lambda^{-1}\},$$

where $\lambda = e^{2\pi}$. The section map Θ' can be written

$$\Theta'((v_1, v_2), z) \equiv ((\lambda v_1, \lambda^{-1} v_2), z).$$

The maximal invariant set for Θ' in Σ' is the set $I' \equiv S^3 \cap \Sigma'$.

5.11 *Proposition.* *$\pi : I' \to S^2 - \{s_1\}$ is a diffeomorphism.*

Proof. One checks that the inverse is

$$\pi^{-1}(w, y) = (0, ((2 - y)^{-\frac{1}{2}} w, (2 - y)^{\frac{1}{2}})).$$

Recall the local asymptotic sets A_s^+ and A_u^+ defined in 4.2. Let

$$\alpha_s' \equiv A_s^+ \cap \Sigma',$$
$$\alpha_u' \equiv A_u^+ \cap \Sigma'.$$

These sets are the positive branches of the stable and unstable manifolds of I' under iterates of Θ'. Note that

$$\alpha_s' = \{((v_1, v_2), z) \in \Sigma' : v_1 = 0\},$$
$$\alpha_u' = \{((v_1, v_2), z) \in \Sigma' : v_2 = 0\}.$$

We now turn our attention to the section map ψ which describes the flow outside the block N. Follow the flow backward from p until it hits Σ' at h' and forward from $\psi(p)$ until it hits Σ' at k'. From the properties of the flow in N, we see that these intersections must occur in time not exceeding 2π. That is, there exist t^-, $t^+ \in (0, 2\pi]$ such that

$$h' \equiv \varphi(p, -t^-) \in \alpha_u' \subset \Sigma',$$
$$k' \equiv \varphi(\psi(p), t^+) \in \alpha_s' \subset \Sigma'.$$

Proposition 4.4 implies that there exist section maps $(\psi^-, T^-, \mathscr{D}^-)$ from Σ' to Σ_u and $(\psi^+, T^+, \mathscr{D}^+)$ from Σ_s to Σ' such that $T^-(h') = t^-$ and $T^+(\psi(p)) = t^+$. By Proposition 5.3, we have $\pi \circ \psi^- = \pi = \pi \circ \psi^+$. Restrict $\mathscr{D}$, $\mathscr{D}^-$, and $\mathscr{D}^+$, if necessary, so that $\psi^-(\mathscr{D}^-) = \mathscr{D}$ and $\psi(\mathscr{D}) = \mathscr{D}^+$, and relabel $\mathscr{D}' \equiv \mathscr{D}^-$. Now define

$$\Psi' : \mathscr{D}' \to \Sigma' \text{ by } \Psi' = \psi^+ \circ \psi \circ \psi^-,$$

and define

$$T' : \mathscr{D}' \to \mathbb{R} \text{ by } T' = T^+ + T + T^-.$$

Then $(\Psi', \mathrm{T}', \mathscr{D}')$ is a section map from Σ' to Σ'. Note that $\Psi'(h') = k'$, and that $\pi(h') = \pi(k') = \pi(p) = s_0$.

Now define

$$L_u' \equiv (\Psi')^{-1}(\alpha_s' \cap \Psi'(\mathscr{D}'))$$
$$L_s' \equiv \Psi'(\alpha_u' \cap \mathscr{D}')$$
$$G_u' \equiv \alpha_u' \cap L_u'$$
$$G_s' \equiv \alpha_s' \cap L_s'.$$

One easily checks that

$$\psi^-(L_u') = \Lambda_u, \ \psi^-(G_u') = \Gamma_u,$$
$$\psi^+(\Lambda_s) = L_s', \ \psi^+(\Gamma_s) = G_s'.$$

Since ψ^- and ψ^+ are diffeomorphisms, all the properties of homoclinic points can be stated in terms of L_u', L_s', G_u' and G_s'. For example, if p is a non-degenerate biasymptotic point, then L_u' and α_u' intersect transversally, L_s' and α_s' intersect transversally, and G_u' and G_s' are two dimensional manifolds.

For the remainder of this section, we assume that p is a transverse homoclinic point. The above comments, together with 5.7 and 5.11 imply:

5.12 ***Proposition.*** *If p is a transverse homoclinic point, then*

$$\pi : G_u' \to S_u, \text{ and}$$
$$\pi : G_s' \to S_s$$

are diffeomorphisms.

Also, 5.8 and 5.11 imply that the following diagram commutes

5.13
$$\begin{array}{ccc} \Gamma'_u & \xrightarrow{\Psi'} & \Gamma'_s \\ \pi \downarrow & & \downarrow \pi \\ S_u & \xrightarrow{\chi} & S_s \end{array}$$

We now place a coordinate system on Σ' which will facilitate our computations in the next section. Make the standard identification of $\mathbb{R}^2$ with $\mathbb{C}$ and let $L : \mathbb{R}^2 \to \mathbb{C}$ be a real linear isomorphism. We shall specify L below, but for now it is arbitrary. Let

$$\Sigma \equiv \{((v_1, v_2), u) \in \mathbb{R}^2 \times \mathbb{R}^2 : |v_1| < 1, |v_2| < 1, 4 - 2v_1v_2 - |Lu|^2 > 0\}.$$

Define

$$\sigma : \Sigma \to \Sigma' : (v, u) \to (v, (z_1, z_2)),$$

where $z_1 = Lu$, and $z_2 = (4 - 2v_1v_2 - |Lu|^2)^{\frac{1}{2}}$. One can check that σ is a diffeomorphism.

The sets I', α'_s, α'_u, $\mathcal{D}'$, L'_u, L'_s, G'_u, and G'_s, together with the points h' and k', when transformed by σ^{-1} to Σ, will be denoted by the same symbol without the prime (see Figure 5.1).

The section map Θ' in the new coordinates becomes $\Theta = \sigma^{-1}\Theta'\sigma$ and can be written

5.14
$$\Theta((v_1, v_2), u) = ((\lambda v_1, \lambda^{-1}v_2), u).$$

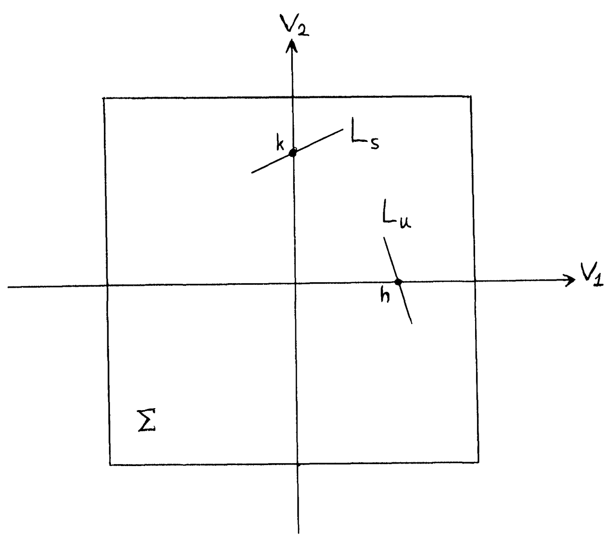

Figure 5.1

The section map Ψ' transforms to Ψ with domain $\mathscr{D}$, and the following diagram commutes

5.15
$$\begin{array}{ccc} \mathscr{D}' & \xrightarrow{\Psi'} & \Sigma' \\ \sigma \uparrow & & \uparrow \sigma \\ \mathscr{D} & \xrightarrow{\Psi} & \Sigma \end{array} .$$

We now wish to establish that the projection map π on Σ' can be thought of as projection onto the last two coordinates of Σ. To this end, define

$$\rho : I \to \mathbb{R}^2 : (v, u) \to u, \text{ and define}$$
$$\mathscr{U} \equiv \rho(I) = \{u \in \mathbb{R}^2 : |Lu| < 2\}.$$

Since $I = \{(v, u) \in \Sigma : v = 0\}$, we see that $\rho : I \to \mathscr{U}$ is a diffeomorphism. Therefore Proposition 5.11 implies the existence of a diffeomorphism α which makes the following diagram commute

5.16
$$\begin{array}{ccc} I & \xrightarrow{\sigma} & I' \\ \rho \downarrow & & \downarrow \pi \\ \mathscr{U} & \xrightarrow{\alpha} & S^2 - \{s_1\}. \end{array}$$

In fact, one can compute that

$$\alpha(u) = \left((4 - |Lu|^2)^{\frac{1}{2}} Lu,\ |Lu|^2 - 2\right),$$

and therefore we have

5.17 $$\alpha(0) = s_0 \quad \text{and} \quad D\alpha(0)\dot{u} = (2L\dot{u}, 0).$$

Now extend ρ to Σ by the formula

$$\rho((v_1, v_2), u) = \left(1 - \frac{1}{2} v_1 v_2\right)^{-\frac{1}{2}} u.$$

From the definition of π and from 5.14 one can see that the following diagram commutes:

5.18
$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma' \\ \rho \downarrow & & \downarrow \pi \\ \mathscr{U} & \xrightarrow{\alpha} & S^2 - \{s_1\}. \end{array}$$

Proposition 5.11 now implies:

5.19 *Proposition.* *If p is a transverse homoclinic point, then $\rho : G_u \to \rho(G_u) \subset \mathscr{U}$ and $\rho : G_s \to p(G_s) \subset \mathscr{U}$ are diffeomorphisms.*

We now transform the orbit map χ by the diffeomorphism α to obtain the new orbit map X. Thus the following diagram commutes

5.20
$$\begin{array}{ccc} \mathcal{S}_u & \xrightarrow{\chi} & S^2 - \{s_1\} \\ \alpha \uparrow & & \uparrow \alpha \\ \alpha^{-1}(\mathcal{S}_u) & \xrightarrow{X} & \mathcal{U} \end{array} .$$

By combining diagrams (5.13), (5.15), (5.18), and (5.20), we obtain the following commutative diagram of diffeomorphisms

5.21
$$\begin{array}{ccc} G_u & \xrightarrow{\Psi} & G_s \\ \rho \downarrow & & \downarrow \rho \\ \mathcal{S} & \xrightarrow{X} & \mathcal{U} \end{array}$$

where $\mathcal{S} \equiv \alpha^{-1}(\mathcal{S}_u) = \rho(G_u)$. Letting

$$A \equiv D\Psi(h), \qquad K \equiv DX(0)$$

and differentiating diagram 5.21 we obtain the following commutative diagram

5.22
$$\begin{array}{ccc} T_hG_u & \xrightarrow{A} & T_kG_s \\ \rho \downarrow & & \downarrow \rho \\ \mathbb{R}^2 & \xrightarrow{K} & \mathbb{R}^2 \end{array} .$$

Denoting $A \equiv [a_{ij}]$ and $A^{-1} \equiv [b_{ij}]$, we have the following:

5.23 *Proposition.* *If p is a transverse homoclinic point, then $a_{11} \neq 0$ and $b_{22} \neq 0$.*

Proof. We prove only that $a_{11} \neq 0$. The proof for b_{22} is similar.
Note first that

$$T_hL_u = \{((v_1, v_2), (u_1, u_2)) : a_{11}v_1 + a_{12}v_2 + a_{13}u_1 + a_{14}u_2 = 0\},$$

$$T_h\alpha_u = \{((v_1, v_2), (u_1, u_2)) : v_2 = 0\}.$$

Since p is a transverse homoclinic point, L_u and α_u intersect transversally at h. Therefore

$$\hat{A} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

has rank 1. Since $G_u = L_u \cap \alpha_u$, we have that

$$T_hG_u = \{(v, u) : \hat{A}(v, u) = 0\}.$$

By Proposition (5.19), $\rho : T_h G_u \to \mathbb{R}^2$ is an isomorphism. Hence the map $(\hat{A}, \rho) : \mathbb{R}^4 \to \mathbb{R}^2 \times \mathbb{R}^2$ has trivial null space. But this map is represented by the matrix

$$(\hat{A}, \rho) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence $a_{11} \neq 0$, and the proof is complete.

Differentiating 5.20, we obtain the diagram

5.23
$$\begin{array}{ccc} T_{s_0}S^2 & \xrightarrow{D\chi(s_0)} & T_{s_0}S^2 \\ {\scriptstyle D\alpha(0)} \uparrow & & \uparrow {\scriptstyle D\alpha(0)} \\ \mathbb{R}^2 & \xrightarrow{K} & \mathbb{R}^2 \end{array} .$$

Since $T_{s_0}S^2 = \{(\dot{w}, \dot{y}) \in \mathbb{C} \times \mathbb{R} : \dot{y} = 0\}$, we may represent $D\chi(s_0)$ as

$$D\chi(s_0)\dot{w} = (K'\dot{w}, 0),$$

where $K' : \mathbb{C} \to \mathbb{C}$ is a real linear isomorphism and may be thought of as a 2×2 matrix. By (5.17),

$$K = L^{-1}K'L.$$

If we assume that p is a hyperbolic homoclinic point, then K' will have two distinct real eigenvalues, one with absolute value greater than one, the other with absolute value less than one. We write these eigenvalues $s_1\mu_1^2$ and $s_2\mu_2^2$ where $\mu_1 > 1 > \mu_2 > 0$ and $s_1 = \pm 1$, $s_2 = \pm 1$. Hence we can choose L so that

5.24
$$K = \begin{bmatrix} s_1\mu_1^2 & 0 \\ 0 & s_2\mu_2^2 \end{bmatrix}.$$

§6 Oscillation and capture. In this section we prove the existence of oscillatory and capture orbits for a flow φ which is a perturbation of the model flow provided that there exists a hyperbolic homoclinic orbit of φ. First we define a sequence of sets $B_n \subset \Sigma$ which we call windows and we assume for each n that $w_n : B_n \to \Sigma$ is a homeomorphism. In 6.4 we give sufficient conditions so that there exists a sequence of points $x_n \in B_n$ such that $w_n(x_n) = x_{n+1}$ for each n. We apply this result to construct oscillatory and capture orbits by constructing sequences of windows and defining w_n to be either Θ or ψ restricted to B_n. We find a sequence of points $x_n \in B_n$ such that $w_n(x_n) = x_{n+1}$ for each n. Each of the points $\sigma(x_n)$ lies on the same orbit of φ and we show, depending on the choice of the windows, that this is either an oscillatory or a capture orbit.

6.1 *Notation:* For $\xi = (\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$, let

$$\|\xi\| \equiv \max\{|\xi_1|, \cdots, |\xi_k|\}$$
$$I^k \equiv \{\xi \in \mathbb{R}^k : \|\xi\| \le 1\}$$
$$\partial I^k \equiv \{\xi \in \mathbb{R}^k : \|\xi\| = 1\}.$$

For our purposes,

$$\xi \in I^4,\ \xi = (\xi_1, \xi_2, \xi_3, \xi_4) = (\xi^1, \xi^2)$$

where $\xi^1 \equiv (\xi_1, \xi_2)$ and $\xi^2 \equiv (\xi_3, \xi_4)$.

Define

$$\partial^- I^4 = \{(\xi^1, \xi^2) \in I^4 : \|\xi^1\| = 1\},$$
$$\partial^+ I^4 = \{(\xi^1, \xi^2) \in I^4 : \|\xi^2\| = 1\}.$$

6.2 ***Definition.*** A *window* is a triple (B, b^-, b^+), where $B \subset \Sigma$, $b^-, b^+ \subset B$, such that there exists a homeomorphism $\beta : (I^4, \partial^- I^4, \partial^+ I^4) \to (B, b^-, b^+)$, called a *standard window homeomorphism* from I^4 to B.

If (B, b^-, b^+) is a window we may drop the b^- and b^+ and simply refer to B as the window. In such a case, we shall write the standard window homeomorphism β as simply $\beta : I^4 \to B$, understanding that $\beta(\partial^- I^4) = b^-$ and $\beta(\partial^+ I^4) = b^+$.

Suppose that (B, b^-, b^+) is a window and $w : B \to w(b) \subset \Sigma$ is a homeomorphism. By $w(B)$ we mean the window $(w(B), w(b^-), w(b^+))$. Note that, if $\beta : I^4 \to B$ is a standard window homeomorphism, then $w \circ \beta : I^4 \to w(B)$ is also a standard window homeomorphism.

6.3 ***Definition.*** Window B is said to be *correctly aligned* with window C if there exist standard window homeomorphisms $\beta : I^4 \to B$ and $\gamma : I^4 \to C$ and an embedding $\nu : I^2 \times I^2 \to I^2 \times I^2$ such that

$$\beta(\eta^1, \eta^2) = \gamma(\xi^1, \xi^2) \Leftrightarrow (\eta^1, \xi^2) = \nu(\xi^1, \eta^2).$$

6.4 ***Theorem.*** *For each integer n let $\{B_n, b_n^+, b_n^-\}$ be a window and let $w_n : B_n \to w_n(B_n)$ be a homeomorphism such that $w_n(B_n)$ is correctly aligned with B_{n+1}. Then for each n there exists $x_n \in B_n$ such that $w_n(x_n) = x_{n+1}$.*

In Section 9 below we give more general definitions of window and correct alignment. Theorem 6.4 is a special case of Theorem 9.6 as is shown in Corollary 9.8. We now proceed to the construction of the windows which will be used to prove oscillation and capture.

Assume that p is a hyperbolic homoclinic point and that L is chosen so that (5.24) holds. Recall that $A = D\Psi(h)$. Assume that

$$\Psi(x) = k + A(x - h) \quad \text{for} \quad x \in \mathscr{D}.$$

We make this assumption to simplify the computations presented below. In view of our hypotheses on the structure of the flow in N, this assumption is relatively harmless.

Write

$$(a_1, a_2, a_3, a_4) \equiv A(1, 0, 0, 0)$$
$$(b_1, b_2, b_3, b_4) \equiv A^{-1}(0, 1, 0, 0).$$

Proposition (5.23) implies that $a_1 \neq 0$ and $b_2 \neq 0$. Introduce the four vectors:

$$w_1 = (1, 0, 0, 0)$$
$$w_2 = (b_5, 0, \mu_1^{-1}, 0)$$
$$w_3 = (b_1, b_2, b_3, b_4)$$
$$w_4 = \mu_2^{-1}(b_6, 0, 0, 1).$$

Since $b_2 \neq 0$, these vectors form a basis for $\mathbb{R}^4$. Choose b_5 and b_6 so that w_2 and w_4 are in $T_h G_u$. Then (5.22) and (5.24) imply that

$$Aw_2 = \mu_1(0, a_5, s_1, 0),$$
$$Aw_4 = (0, a_6, 0, s_2\mu_2).$$

where a_5 and a_6 are determined by the relation $Aw_2, Aw_4 \in T_k G_s$. We also have that

$$Aw_1 = (a_1, a_2, a_3, a_4), \quad \text{and} \quad Aw_3 = (0, 1, 0, 0).$$

Note that h and k can be written

$$h = (\tilde{h}, 0, 0, 0), \quad \text{and} \quad k = (0, \tilde{k}, 0, 0),$$

where $0 < \tilde{h} < 1$, $0 < \tilde{k} < 1$. Choose Δ so that

$$\tilde{h} < \Delta < 1, \qquad \tilde{k} < \Delta < 1.$$

Then choose $\delta > 0$ so that

$$\delta(1 + |b_5| + |b_6|) < 1 - \tilde{h}\Delta^{-1}, \quad \text{and}$$
$$\delta(1 + |a_5| + |a_6|) < 1 - \tilde{h}\Delta^{-1}.$$

Finally, choose an integer $N > 0$ so that all eight of the following inequalities hold

(6.5)
$$\lambda^{-2N} < \delta|b_2|$$
$$|b_4|\lambda^{-2N} < \delta|b_2|(\mu_2^{-1} - 1)$$
$$|b_3|\lambda^{-2N} < \delta|b_2|(1 - \mu_1^{-1})$$
$$|b_1 - b_4b_6|\lambda^{-2N} < |b_2|[1 - \tilde{h}\Delta^{-1} - \delta(1 + |b_5| + |b_6|)]$$
$$\lambda^{-2N} < \delta|a_1|$$

(6.6)
$$|a_3|\lambda^{-2N} < \delta|a_1|(\mu_1 - 1)$$
$$|a_4|\lambda^{-2N} < \delta|a_1|(1 - \mu_2)$$
$$|a_2 - s_1 a_3 a_5|\lambda^{-2N} < |a_1|[1 - \tilde{k}\Delta^{-1} - \delta(1 + |a_5| + |a_6|)].$$

We now construct $2N + 1$ windows $B^{-N}, \cdots B^{-1}, B^0, B^1, \cdots B^N$ as follows:

$$B^k \equiv \{((v_1, v_2), u) \in \Sigma : |v_1| \le \Delta\lambda^{k-N}, |v_2| \le \Delta\lambda^{-N-k}, \|u\| \le \delta\Delta\}.$$

A standard window homeomorphism for each of these is given by

$$\beta^k : I^4 \to B^k$$
$$\beta^k(\eta_1, \eta_2, \eta_3, \eta_4) \equiv \Delta(\lambda^{k-N}\eta_1, \lambda^{-N-k}\eta_3, \delta\eta_2, \delta\eta_4).$$

Recall the section map Θ given by 5.14 and note that $\Theta(B^k) = B^{k+1}$.

We now construct an addition window $B^* \equiv \beta^*(I^4)$ by defining a standard window homeomorphism $\beta^* : I^4 \to \Sigma$ by

$$\beta^*(\xi_1, \xi_2, \xi_3, \xi_4) \equiv h + \delta\Delta(\xi_1 w_1 + \xi_2 w_2 + \xi_3 w_3 + \xi_4 w_4).$$

6.7 *Proposition.* *B^N is correctly aligned with B^*.*

Proof. An elementary computation shows that $\beta^N(\eta) = \beta^*(\xi)$ if and only if

$$\eta_1 = \tilde{h}\Delta^{-1} + \delta(\xi_1 + b_5\xi_2 + b_6\eta_4) + b_2^{-1}\lambda^{-2N}(b_1 - b_4 b_6)\eta_3$$
$$\eta_2 = \mu_1^{-1}\xi_2 + (\delta b_2)^{-1}\lambda^{-2N} b_3\eta_3$$
$$\xi_3 = (\delta b_2)^{-1}\lambda^{-2N}\eta_3$$
$$\xi_4 = \mu_2(-(\delta b_2)^{-1}\lambda^{-2N} b_4\eta_3 + \eta_4).$$

These equations define the function $\nu(\xi_1, \xi_2, \eta_3, \eta_4)$. By inspection, ν is a linear isomorphism plus a constant. Estimates (6.5) show that $\nu(I^4) \subset I^4$ and the proof is complete.

6.8 *Proposition.* *$\Psi(B^*)$ is correctly aligned with B^{-N}.*

Proof. Another elementary computation shows that $(\Psi \circ \beta^*)(\xi) = \beta^{-N}(\eta)$ if and only if

$$\xi_1 = (\delta a_1)^{-1}\lambda^{-2N}\eta_1$$
$$\xi_2 = s_1\mu_1^{-1}(-(\delta a_1)^{-1}\lambda^{-2N} a_3\eta_1 + \eta_2)$$
$$\eta_3 = \tilde{k}\Delta^{-1} + \delta(\xi_3 + s_1 a_5\eta_2 + \delta a_6\xi_4) + a_1^{-1}\lambda^{-2N}(a_2 - s_1 a_3 a_5)\eta_1$$
$$\eta_4 = s_2\mu_2\xi_4 + (\delta a_1)^{-1}\lambda^{-2N} a_4\eta_1.$$

As above, estimates (6.6) imply that these equations define an embedding $I^4 \to I^4$, and the proof is complete.

6.9 *Proposition.* *For $-N \le k \le N - 1$, $\Theta(B^k)$ is correctly aligned with B^k.*

Proof. $\Theta \circ \beta^k(\eta) = \beta^k(\xi)$ if and only if

$$\eta_1 = \lambda^{-1}\xi_1, \qquad \eta_2 = \xi_2, \qquad \xi_3 = \lambda^{-1}\eta_3, \quad \text{and} \quad \xi_4 = \eta_4.$$

These equations define an embedding $I^4 \to I^4$, and the proof is complete.

Remark. Note that, since $\Theta(B^k) = B^{k+1}$, $\Theta(B^k)$ and B^{k+1} are correctly aligned.

6.10 *Lemma.* *Given any bi-infinite sequence of integers m_k, $k \in \mathbb{Z}$, with $m_k \geq 2N + 1$, there exist points $x_k \in B^*$, such that*

$$\Theta^{m_k - 1}(\Psi(x_k)) = x_{k+1}.$$

Proof. Define M_k for $k \in \mathbb{Z}$, by

$$M_0 = 0, \qquad M_k + m_k = M_{k+1}.$$

Then let

$$B_{M_k + j} = \begin{Bmatrix} B^* & \text{if } j = 0 \\ B^{-N} & \text{if } j = 1, \cdots, m_k - 2N \\ B^{N + j - m_k} & \text{if } j = m_k - 2N + 1, \cdots, m_k - 1 \end{Bmatrix}$$

$$w_{M_k + j} = \begin{Bmatrix} \Psi & \text{if } j = 0 \\ \Theta & \text{if } j = 1, \cdots, n_k - 1 \end{Bmatrix}.$$

By Propositions 6.7, 6.8, and 6.9, B_n, w_n satisfy the hypotheses of Theorem 6.4. Hence there exist $y_n \in B_n$ such that $w_n(y_n) = y_{n+1}$. Letting $x_k = y_{M_k}$ completes the proof.

As discussed in Moser's book [9], one can also consider finite or semi-infinite sequences $\{m_k\}$ beginning or ending with the symbol ∞. If, for some $k' \geq 0$, $m_{k'} = \infty$, while $m_k < \infty$, $k < k'$, then define

$$B_j = B^{-N}, \quad \text{if} \quad j > M_{k'},$$

$$w_j = \Theta, \quad \text{if} \quad j > M_{k'}.$$

One then concludes that $\Theta^{m_k - 1}(\Psi(x_k)) = x_{k+1}$ for $k < k'$, while $\Theta^j(\Psi(x_{k'})) \in \Sigma$ for every $j \geq 0$. Therefore $\Psi(x_{k'}) \in \alpha_s$, and the orbit through $\sigma(x_{k'})$ is asymptotic to S^3 as $t \to \infty$. If, for some $k' < 0$, $m_{k'} = \infty$, while $m_k < \infty$, $k > k'$, then define

$$B_j = B^{N-1}, \quad \text{if} \quad j < M_{k'+1}$$

$$w_j = \Theta, \quad \text{if} \quad j < M_{k'+1}.$$

One then concludes that $\Theta^{m_k - 1}(\Psi(x_k)) = x_{k+1}$ for $k > k'$, while $\Theta^{-j}(x_{k'+1}) \in \Sigma$ for every $j \leq 0$. Therefore $x_{k'+1} \in \alpha_u$, and the orbit through $\sigma(x_{k'+1})$ is asymptotic to S^3 as $t \to -\infty$. Note that, if the sequence begins and ends with ∞, then the corresponding orbit is biasymptotic.

6.11 *Theorem.* *If φ is a perturbation of the model flow which has a hyperbolic homoclinic orbit, then it has oscillatory orbits.*

Proof. Choose m_k with $2N + 1 \leq m_k < \infty$, so that $\limsup_{k\to\infty} m_k = \infty$. Let x_k be given by Lemma 6.10. Then, for arbitrarily large values of t, the orbit through $\sigma(x_0)$ stays in N for an arbitrarily large interval of time. Therefore $\omega(\sigma(x_0)) \cap S^3 \neq \emptyset$. Since the orbit through $\sigma(x_0)$ leaves N infinitely often, $\omega(v(x_0)) \cap M - N \neq \emptyset$. By Definition 3.13, the orbit through $\sigma(x_0)$ is oscillatory, and the proof is complete.

6.12 *Theorem.* *If φ has a hyperbolic homoclinic orbit, then it has capture orbits.*

Proof. Fix $M \geq 2N + 1$. Choose m_k so that $2N + 1 \leq m_k \leq M$ for $k > 0$ and let $m_0 = \infty$. Let x_k be given by Lemma 6.10. Since $m_0 = \infty$, $\alpha(\sigma(x_0)) \in S^3$. Since $m_k \leq M$ for $k > 0$, the orbit through $\sigma(x_0)$ remains bounded away from S^3 as $t \to \infty$. Therefore $\omega(\sigma(x_0)) \cap S^3 = \emptyset$, so $\sigma(x_0)$ is a capture orbit by Definition 3.19, and the proof is complete.

§7 **A perturbation theorem.** In this section, we show that the existence of a non-degenerate homoclinic point remains under small perturbations. The new homoclinic point may be asymptotic to a periodic orbit different from the one to which the unperturbed homoclinic point is asymptotic. Before proving the theorem, we develop a characterization of homoclinic points.

For $x = (v, z) \in N$, let $e(x) = (0, iz) \in \mathbb{R}^2 \times C^2$. The vector field e has the following properties:

7.1 $$e(x) \in T_xM \quad \text{for each} \quad x \in N \quad \text{and}$$

7.2 $$e(x) \in \ker(D\pi(x)) \quad \text{for each} \quad x \in N.$$

7.3 Let $x \in a_u$. Then $e(x) \in T_xa_u$, and $D\pi(x)|T_xa_u$ is surjective.

7.4 Let $x \in a_s$. Then $e(x) \in T_xa_s$, and $D\pi(x)|T_xa_s$ is surjective.

As a consequence of the above properties, we have:

7.5 *Proposition.* *Let $p \in a_u$ be a homoclinic point, let ψ be given by Lemma 4.5, and let $q = \psi(p)$. Then p is a transverse homoclinic point if and only if*
(a) $e(p) \notin T_p\Lambda_u$, and
(b) $e(q) \notin T_q\Lambda_s$.

We are now ready to prove a perturbation result. Consider the following differential equation

7.6 $$\dot{x} = F^0(x) + f_\epsilon(x)$$

with integral

$$H(x) = H^0(x) + h_\epsilon(x),$$

where, for each ϵ, f_ϵ and h_ϵ satisfy the hypotheses 3.8 and 3.9. Assume that f_ϵ and h_ϵ vary smoothly with ϵ, and let φ_ϵ be the flow on M_ϵ associated with 7.6.

7.7 *Theorem.* *Suppose that $p_0 \in a_u$ is a non-degenerate homoclinic point for φ_0. Then, for each small ϵ, there is a non-degenerate homoclinic point p_ϵ for φ_ϵ. Furthermore, p_ϵ varies smoothly with ϵ.*

Proof. Let $(\psi_0, T, \mathscr{D})$ be the section map from Σ_u to Σ_s obtained by applying Lemma 4.5 to φ_0 and p_0. A standard use of the implicit function theorem tells us that, by possibly restricting to smaller $\mathscr{D}$, there is a section map $\psi_\epsilon : \mathscr{D} \to \Sigma_s$ for the flow φ_ϵ, for small enough ϵ. Furthermore, ψ_ϵ varies smoothly with ϵ. Let

$$\Lambda_u^\epsilon \equiv \psi_\epsilon^{-1}(a_s), \qquad \Lambda_s^\epsilon \equiv \psi_\epsilon(a_u).$$

Since Λ_u^0 and a_u, and Λ_s^0 and a_s, intersect transversally, so do Λ_u^ϵ and a_u, and Λ_s^ϵ and a_s, for small enough ϵ. Let

$$\Gamma_u^\epsilon = \Lambda_u^\epsilon \cap a_u, \qquad \Gamma_s^\epsilon = \Lambda_s^\epsilon \cap a_s.$$

For each ϵ, Γ_u^ϵ and Γ_s^ϵ are two-dimensional manifolds varying smoothly with ϵ. Furthermore, every point on Γ_u^ϵ is a non-degenerate biasymptotic point for φ_ϵ.

Applying Proposition 7.5 to φ_0 and p_0, we have that

$$e(p_0) \notin T_{p_0}\Gamma_u^0.$$

Hence, for small enough ϵ,

$$e(x) \notin T_x\Gamma_u^\epsilon \quad \text{for each} \quad x \in \Gamma_u^\epsilon.$$

Therefore, by 7.2, $D\pi(x)|T_x\Gamma_u^\epsilon$ is an isomorphism. Hence $\gamma_\epsilon \equiv (\pi|\Gamma_u^\epsilon)^{-1}$ exists on some neighborhood S_u of $s_0 \equiv \pi(p_0)$ in S^2. Furthermore, γ_ϵ varies smoothly with ϵ.

Now define

$$\chi_\epsilon : S_u \to S^2, \qquad \chi_\epsilon = \pi \circ \psi_\epsilon \circ \gamma_\epsilon.$$

By hypothesis, $D\chi_0(s_0)$ does not have eigenvalue 1. Hence, by the implicit function theorem, there is an $s_\epsilon \in S_u$ such that $\chi_\epsilon(s_\epsilon) - s_\epsilon$ and $D\chi_\epsilon(s_\epsilon)$ does not have eigenvalue 1. By Proposition 5.6, $p_\epsilon = \gamma_\epsilon(s_\epsilon)$ is a homoclinic point. By construction it is non-degenerate, and the proof is complete.

§8 Hamiltonian perturbations. In this section we show that if φ is a Hamiltonian perturbation of the model flow then φ has a homoclinic orbit. We also show that an arbitrary perturbation of the model flow need not have this property.

8.1 *Proposition.* *Example 4.11 with $\epsilon > 0$ has no homoclinic points.*

Proof. Suppose $p \in a_u$ is a homoclinic point. By arguments similar to those of Proposition 4.9 there exist $n \geq 1$ and real numbers s_j and t_j for $j = 0, 1, \ldots, n - 1$, with $s_0 = 0$ and $s_j < t_j < s_{j+1}$, so that $\varphi(p, [s_j, t_j]) \subset Q$, $\varphi(p, [t_j, s_{j+1}]) \subset N$, and $p' \equiv Q(p, t_{n-1}) \in a_s$.

As in the proof of Proposition 4.13. Write

$$\varphi(p, s_j) = (1, g(z^j), z^j), \quad \text{for} \quad j = 0, 1, \ldots, n - 1, p' = (1, g(z^n), z^n).$$

Thus $y_0 = p$, $y_{j+1} = (\sigma^{-1}\psi\sigma)(y_j)$ and $y_n = p'$. Since p is a homoclinic point $\pi(p) = \pi(p')$. Hence there exists a real number θ such that $e^{i\theta}z^0 = z^n$. Therefore $|z_1^0| = |z_1^n|$ and $|z_2^0| = |z_2^n|$. However, we have

$$|z_1^n| = |z_1^0| \prod_{j=1}^{n} (1 + \epsilon\beta(g(z^j))).$$

Since $\beta(q(z^0)) = 1$ and β is non-negative, the product above must be greater than one. Thus $|z_1^0| = |z_1^n| = 0$ and $|z_2^0| > 0$. The product

$$\prod_{j=1}^{n} (1 + \epsilon\beta(g(z^j)))^{-1}z$$

is less than one. This contradicts the fact that

$$|z_2^n| = |z_2^0| \prod_{j=1}^{n} (1 + \epsilon\beta(g(z^j)))^{-1}$$

and therefore p cannot be a homoclinic point. This completes the proof.

The following theorem is a special case of a theorem proved by J. Moser [10].

8.2 *Theorem.* *Let P be a symplectic manifold and let $K : P \to \mathbb{R}^1$ be a Hamiltonian function with 0 as a regular value. Let $A = K^{-1}(0)$ and let Φ be the flow on P generated by the Hamiltonian vector field associated with K. If $\gamma : P \to P$ is an exact symplectic map which is sufficiently close to the identity, then there exist $p \in A$ and a real number t close to zero such that $\Phi(\gamma(p), t) = p$.*

The proof of this theorem uses ideas which go back to Poincaré and which are developed in a coordinate free way by Weinstein [14]. We use this theorem to prove the following:

8.3 *Theorem.* *If φ is a sufficiently small Hamiltonian perturbation of the model flow, then φ has a homoclinic orbit.*

Proof. Σ_u is a symplectic manifold with 2-form $\Omega|\Sigma_u$ where Ω is the 2-form on $\mathbb{R}^2 \times \mathbb{C}^2$ given by

$$\Omega = dv_1dv_2 + (2i)^{-1}(d\bar{z}_1dz_1 + d\bar{z}_2dz_2).$$

Define $K : \Sigma_u \to R^1$ by $K((1, g(z)), z) = (1/2)\|z\|^2 - 2$. Then $K^{-1}(0) = a_u$. Now let $\psi : \Sigma_u \to \Sigma_s$ denote the section map of φ. Since φ is a Hamiltonian flow the section map ψ is exact symplectic. In particular $\psi^*\Omega|\Sigma_s = \Omega|\Sigma_u$. Since φ is a small perturbation of φ^0, ψ is a small perturbation of ψ^0. Define $\gamma = (\psi^0)^{-1}\psi$. Thus γ is an exact symplectic map of Σ_u which is close to the identity. Therefore by Theorem 8.2 there exists $p \in a_u$ and $t \in R^1$ such that $\varphi_K(\gamma(p), t) = p$ where φ_K is the flow generated by the Hamiltonian vector field associated with K.

It remains to show that p is a homoclinic point. Let $p = ((1, 0), z)$. Then

$$\gamma(p) = \varphi_K(p, t) = ((1, 0), e^{-it}z).$$

Hence $\psi(p) = \psi^0(\varphi_K(p, t)) = \psi^0((1, 0), e^{-it}z)$. It follows that $\pi(\psi(s)) = \pi(\psi^0((1, 0), e^{-it}z)) = \pi(p)$. Therefore by 5.3, p is a homoclinic point. This completes the proof.

§9. A theorem on windows. Let (X, d) be a metric space.

9.1 ***Definition.*** A *window* is a collection (B, b^+, b^-, μ, ν) with B a compact subset of X, with b^+, b^- closed subsets of ∂B such that $\partial B = b^+ \cup b^-$, and with $\mu \in H^k(B, b^+)$ and $\nu \in H^\ell(B, b^-)$ non-trivial Alexander cohomology classes (with integer coefficients) for some k and ℓ.

9.2 ***Definition.*** Let $(B_n, b_n^+, b_n^-, \mu_n, \nu_n)$ for $n = 1, 2$ be windows. Define $B_{12} = B_1 \cap B_2$, $b_{12}^+ = b_1^+ \cap B_2$ and $b_{12}^- = B_1 \cap b_2^-$. Then *window* B_1 *intersects window* B_2 *correctly if* $B_1 \cap b_2^+ \subset b_{12}^+$ and $b_1^- \cap B_2 \subset b_{12}^-$, if $b_{12}^+ \cup (B_2 - B_{12})$ is closed and if $b_{12}^- \cup (B_1 - B_{12})$ is closed. It follows that $\partial B_{12} = b_{12}^+ \cup b_{12}^-$, $b_2^+ \subset b_{12}^+ \cup (B_2 - B_{12})$, and $b_1^- \subset b_{12}^- \cup (B_1 - B_{12})$. Figure 9.1 below shows windows intersecting correctly.

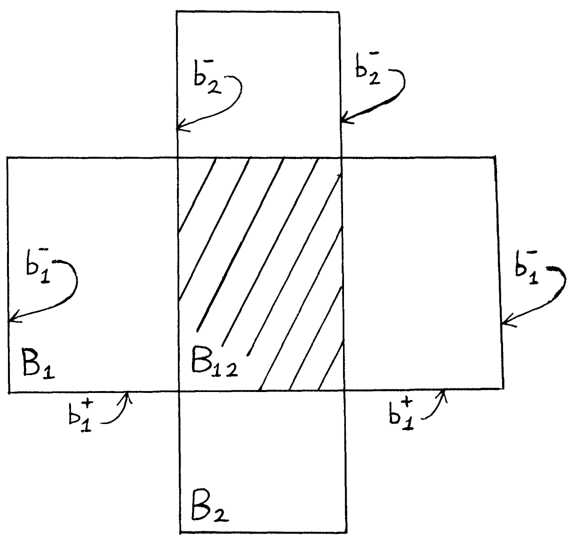

Figure 9.1

9.3 ***Definition.*** Suppose that window B_1 intersects window B_2 correctly. Let $j_1, \cdots, k_3$ denote inclusion maps between the pairs of spaces indicated below.

$$(B_1, b_1^+) \xleftarrow{j_1} (B_{12}, b_{12}^+) \xrightarrow{j_2} (B_2 b_{12}^+ \cup (B_2 - B_1)) \xleftarrow{j_3} (B_2, b_2^+)$$

$$(B_2, b_2^-) \xleftarrow{k_1} (B_{12}, b_{12}^-) \xrightarrow{k_2} (B_1, b_{12}^- \cup (B_1 - B_2)) \xleftarrow{k_3} (B_1, b_1^-).$$

By the strong excision property for Alexander cohomology [15, p. 318] the maps j_2^* and k_2^* are isomorphisms of the appropriate cohomology groups. Define

$$J^* : H^k(B_1, b_1^+) \to H^k(B_2, b_2^+)$$

by $J^* = j_3^*(j_2^*)^{-1}j_1^*$. Define

$$K^* : H^\ell(B_2, b_2^-) \to H^\ell(B_1, b_1^-)$$

by $K^* = k_3^*(k_2^*)^{-1}k_1^*$. Define

$$\mu_{12} = j_1^*(\mu_1) \quad \text{and} \quad \nu_{12} = k_1^*\nu_2).$$

Then $(B_{12}, b_{12}^+, b_{12}^-, \nu_{12}, \mu_{12})$ is a window.

9.4 *Definition.* *Window B_1 correctly aligns with window B_2* if window B_1 intersects window B_2 correctly and $J^*(\mu_1) = \pm\mu_2$ and $K^*(\nu_2) = \pm\nu_1$.

9.5 *Lemma 1.* *Suppose that $(B_n, b_n^+, b_n^-, \mu_n, \nu_n)$ for $n = 1, 2, 3$ are windows such that B_1 correctly aligns with B_2 and B_2 correctly aligns with B_3. Then window B_{12} correctly aligns with window B_{23}.*

Proof. Let $B_{12} \cap B_{23} = B_{123}$. Then

$$B_{12}^- \cap B_{23} = B_1 \cap b_2^- \cap B_3 \subset B_1 \cap B_2 \cap b_3^- = b_{123}^-.$$

$$B_{12} \cap b_{23}^+ = B_1 \cap b_2^+ \cap B_3 \subset b_1^+ \cap B_2 \cap B_3 = b_{123}^+.$$

Thus the first condition that B_{12} intersects B_{23} correctly is satisfied. Now $b_{123}^+ \cup (B_{23} - B_{123}) = [b_{12}^+ \cup (B_2 - B_{12})] \cap B_3$ and thus $b_{123}^+ \cup (B_{23} - B_{123})$ is closed. Similarly $b_{123}^- \cup (B_{12} - B_{123})$ is closed and this verifies that B_{12} intersects B_{23} correctly. To show that $J^*(\mu_{12}) = \pm\mu_{23}$ consider the commutative diagram

$$\begin{array}{ccccccc} H^k(B_{12}, b_{12}^+) & \to & H^k(B_2, b_{12}^+ \cup (B_2 - B_{12})) & \to & H^k(B_2, b_2^+) & \to & H^k(B_{23}, b_{23}^+) \\ & \searrow & & & & \nearrow & \\ & & H^k(B_{123}, b_{123}^+) & \to & H^k(B_{23}, b_{123}^+ \cup (B_{23} - B_{123})) & & \end{array}$$

where all the homomorphisms are induced from inclusion maps or excision isomorphisms. It follows from the definitions of μ_{12}, μ_{23}, and the homomorphisms J^* between appropriate groups, and the commutativity of the above diagram that $J^*(\mu_{12}) = \pm\mu_{23}$. One similarly shows that $K^*(\nu_{23}) = \pm\nu_{12}$. This completes the proof.

9.6 *Theorem.* *Let $(B_n, b_n^+, b_n^-, \mu_n, \nu_n)$ for $n \in Z$ be a biinfinite sequence of windows in X. Suppose for each n that f_n is a homeomorphism of B_n into X such* *that the window $f_n(B_n)$ correctly aligns with B_{n+1}. Suppose that the cup product $\mu_0 \cup \nu_0$ is a non-zero element of $H^{k+\ell}(B_0, \partial B_0)$. Then there exists a sequence of points $x_n \in B_n$ such that $f_n(x_n) = x_{n+1}$ for each n.*

Proof. For $n \geq 0$ define $C^n = \{x \in B_0 : f_k \circ \cdots \circ f_0(x) \in B_{k+1}$ for $k = 1, \cdots, n-1\}$, and $E^n = \{x \in B_0 : f_{-k}^{-1} \circ \cdots \circ f_{-1}^{-1}(x) \in B_{-k}$ for $k = 1, \cdots, n\}$. It is sufficient to show that $C^n \cap E^n \neq \varnothing$ for each $n \geq 0$. Consider the commutative diagram

$$\begin{array}{ccc} H^k(B_0, b_0^+) \times H^\ell(B_0, {}_0^-) & \xleftarrow{\alpha} & H^k(B_0, B_0 - E^n) \times H^k(B_0, B_0 - C^n) \\ \downarrow \cup & & \downarrow \cup \\ H^{k+1}(B_0, \partial B_0) & \xleftarrow{\beta} & H^{k+1}(B_0, B_0 - C^n \cap E^n) \end{array}$$

where the vertical homomorphisms are cup product homomorphisms and the horizontal homomorphisms are induced from inclusion maps. To show that $C^n \cap E^n$ is non-empty it is sufficient to show that $\mu_0 \times \nu_0$ is in the range of α since commutativity of the diagram implies that $\mu_0 \cup \nu_0$ is in the range of β and therefore $H^{k+1}(B_0, B_0 - C^n \cap E^n)$ is non-trivial.

Let the notation c.a.w. stand for the phrase "correctly aligns with". We will show that C^n and E^n are windows such that E^n c.a.w. B_0 and B_0 c.a.w. C^n. Suppose this is true for the moment. Then we have the following commutative diagram

$$\begin{array}{ccccc} H^k(E^n, e^{n+}) & \rightarrow & H^k(B_0, e^{n+} \cup (B_0 - E^n)) & \rightarrow & H^k(B_0, b_0^+). \\ & & \downarrow & \nearrow \alpha_1^* & \\ & & H^k(B_0, B_0 - E^n) & & \end{array}$$

Since E^n c.a.w. B_0 we have μ_0 in the image of $H^k(E^n, e^{n+})$ under the composition of the horizontal homomorphisms. It follows that μ_0 is in the range of α_1^*. A similar argument shows that ν_0 is in the range of $\alpha_2^* : H^\ell(B_0, B_0 - C^n) \rightarrow H^\ell(B_0, b_0^-)$ where α_2 is the inclusion map of (B_0, b_0^-) into $(B_0 B_0 - C^n)$. Therefore $\mu_0 \times \nu_0$ is in the image of $\alpha = (\alpha_1^*, x_2^*)$.

It remains to show that E^n and C^n are correctly aligned windows. From its definition, $E^1 = f_{-1}(B_{-1}) \cap B_0$. We have $f_{-1}(B_{-1})$ c.a.w. B_0 and B_0 c.a.w. B_0. Hence by Lemma 1 E^1 is a window and E^1 c.a.w. B_0. Define $E_{-1} = f_{-1}^{-1}(E^1)$. We have $f_{-1}(B_{-1})$ c.a.w. $f_{-1}(B_{-1})$ and $f_{-1}(B_{-1})$ c.a.w. B_0. Hence by Lemma 1 $f_{-1}(B_{-1})$ c.a.w. E^1 and therefore B_{-1} c.a.w. E_{-1}. Define $E_{-\ell} = f_{-\ell}^{-1} \circ \cdots \circ f_{-1}^{-1}(E^\ell)$ and as an induction hypothesis suppose that $E^{\ell-1}$ c.a.w. B_0 and $B_{-\ell+1}$ c.a.w. $E_{-\ell+1}$. Then $f_{-\ell}(E_{-\ell}) = f_{-\ell}(B_{-\ell}) \cap E_{-\ell+1}$. We have $f_{-\ell}(B_{-\ell})$ c.a.w. $B_{-\ell+1}$ and $B_{-\ell+1}$ c.a.w. $E_{-\ell+1}$. Hence by Lemma 1, $f_{-\ell}(E_{-\ell})$ is a window and $f_{-\ell}(E_{-\ell})$ c.a.w. $E_{-\ell+1}$. Therefore $E_{-\ell} + f_{-\ell}^{-1}(E_{-\ell})$ is a window and also E^ℓ c.a.w. $E^{\ell-1}$ since this pair of windows is the homeomorphic image of the pair $f_{-\ell}(E_{-\ell})$ and $E_{-\ell+1}$. By induction we have $E^{\ell-1}$ c.a.w. B_0. Hence by Lemma 1 E^ℓ c.a.w. B_0. Since $f_{-\ell}(B_{-\ell})$ c.a.w. $B_{-\ell+1}$ and by induction $B_{-\ell+1}$ c.a.w. $E_{-\ell+1}$ Lemma 1 implies that $f_{-\ell}(B_{-\ell})$ c.a.w. $f_{-\ell}(E_{-\ell})$ and therefore $B_{-\ell}$ c.a.w. $E_{-\ell}$. This completes the induction argument showing that E^n is a window such that E^n c.a.w. B_0. A similar argument shows that C^n is a window such that B_0 c.a.w. C^n. This completes the proof.

9.7 Proposition. *If* $(B_n, b_n^+, b_n^-, \mu_n, \nu_n)$ *are windows for* $n = 1, 2$ *such that* B_1 *correctly aligns with* B_2 *according to the definition of* §6, *then* B_1 *correctly aligns with* B_2 *under the present definition.*

Proof. Let $\beta_n : I^4 \to B_n$ be the standard window homeomorphisms of §6 for $n = 1, 2$. Then using the notation of §6 there exists an imbedding $\gamma : I^4 \to I^4$ such that $\beta_1(\xi^1, \xi^2) = \gamma(\eta^1, \eta^2)$ if and only if $(\xi^1, \eta^2) = \gamma(\eta^1, \xi^2)$. To show that $b_1^- \cap B_2 \subset b_2^-$ suppose that $\beta_1(\xi^1, \xi^2) \in b_1^- \cap B_2$. Also suppose that $|\xi^2| < 1$. Then $\beta_1(\xi^1, \xi^2) = \beta_n(\eta^1, \eta^2)$ and $|\xi^1| = 1$. Since $(\xi^1, \eta^2) = \gamma(\eta^1, \xi^2)$ and γ is an imbedding it follows that $(\eta^1, \xi^2) \in \partial I^4$. Therefore $|\eta^1| = 1$ and $\beta(\xi^1, \xi^2) \in b_2^-$. Since b_2 is closed it follows from the continuity of β_1 that $\beta_1(\xi^1, \xi^2) \in b_2^-$ provided that $|\xi^1| = 1$ and $\beta_1(\xi^1, \xi^2) \in B_2$. Hence $b_1^- \cap B_2 \subset b_2^-$. Similarly it can be shown that $B_1 \cap b_2^+ \subset b_1^+$.

Next we must show that $b_{12}^+ \cup (B_2 - B_{12})$ is closed. Suppose not. Then there exists a limit point q of this set which does not belong to it. Hence q is a limit point of $B_2 - B_{12}$ and it follows that $q \in \partial(B_{12}) = b_{12}^+ \cup b_{12}^-$. Since $q \notin b_{12}^+$ choose a convex neighborhood W of q in B_2 such that $W \cap b_2^+ = \varnothing$. Since q is a limit point of $B_2 - B_{12}$ there exists a point $s \in W$ with $r \notin B_1$. Since $q \in \partial B_{12}$ there exists a point $s \in W \cap \text{int } B_{12}$. The line segment joining r and s must intersect ∂B_1, at an interior point of B_2. By the choice of W this intersection point belongs to b_1^- contradicting the fact that $b_1^- \cap B_2 \subset b_2^-$. Therefore $b_{12}^+ \cup (B_2 - B_{12})$ is closed. Similarly one shows that $b_{12}^- \cup (B_1 - B_{12})$ is closed and hence B_1 intersects B_2 correctly.

It remains to show that $J^*(\mu_1) = \pm\mu_2$ and $K^*(\mu_2) = \pm\nu_1$. The groups $H^2(B_n, b_n^+)$ and $H^2(B_n, b_n^-)$ are isomorphic to the integers. Thus it is sufficient to show that J^* and K^* are isomorphisms. The inclusion map $i_1 : (B_{12}, b_{12}^+) \to (B_1, b_1^+)$ is a relative homeomorphism and hence j_1 is an isomorphism [ ]. j_2^* is known to be an isomorphism by the strong excision property of Alexander cohomology. It follows from the five lemma applied to the exact sequences of the pairs (B_2, b_2^+) and $(B_2, b_{12}^+ \cup (B_2 - B_{12}))$ that j_3^* is an isomorphism provided that the inclusion map of b_2^+ into $b_{12}^+ \cup (B_2 - B_{12})$ induces an isomorphism at the cohomology level. Thus it is sufficient to show that b_2^+ is a deformation retract of $b_{12}^+ \cup (B_2 - B_{12})$.

Choose $q_0 \in \text{int } B_{12}$ say $q_0 = \beta(\xi_0^1, \xi_0^2)$. Define $R = \{q \in B_{12} : q = \beta_1(\xi^1, \xi_0^2)$ for some $\xi^1 \in I^k\}$. $\beta_2^{-1}(R)$ is the graph of a continuous function $g : I^2 \to I^2$ where $g(\eta^1) = \pi_2 \circ \gamma(\xi^1, \xi_0^2)$, and where $\pi_2(\xi^1, \eta^2) = \eta^2$ defines the projection π_2. It follows that b_2^- is a deformation retract of $B_2 - R$. It also follows from the definition of R that $b_{12}^+ \cup (B_2 - B_{12}) \subset B_2 - R$ and therefore b_2^- is a deformation retract of $b_{12}^+ \cup (b_2 - B_{12})$. This completes the proof.

9.8 Corollary. *Theorem 6.4 now follows from Theorem 9.6 and the above proposition.*

Remarks. The definition of correct alignment given in Section 6 does not have the transitivity property of Lemma 1. However if the standard window homeomorphisms β_1 and β_2 have range R^4 and satisfy certain Lipshitz conditions then Lemma 1 can be established and Theorem 9.6 can be proven without using algebraic topology. For a result in this spirit see [4].

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The first author was supported by NSF Grant MCS 76-84420. The second author was supported by NSF Grant MCS 78-00907.

Received February 6, 1978

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