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Von Zeipel's theorem on singularities in celestial mechanics

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1 Introduction

A solution of a system of ordinary differential equations is said to experience a *singularity* at time $t^* < \infty$ if the solution cannot be extended beyond t^*. A full understanding of the nature of the singularities which can arise in solutions of the n-body problem of classical celestial mechanics has eluded mathematics to this day. Interest in these singularities seems to have originated with Painlevé in 1895 [6], when he asked whether all singularites are due to collisions between the particles.

An important step towards an answer to Painlevé's question was taken by Hugo von Zeipel in 1908 [15]. He showed that, if the positions of all the particles remain bounded as t approaches t^*, then the singularity must be due to a collision. In other words, a noncollision singularity can occur only if the system of particles becomes unbounded in finite time.

Von Zeipel's paper fell into obscurity for a number of years, and recent references to it allude to "gaps" and "errors" in the proof [14, p. 431; 10, p. 15; 8, p. 312]. Although von Zeipel's four page paper is briefer than one might ideally like, it contains all the essential ingredients of a complete proof. Indeed, when taken in hostorical context, it contains impressive insights.

The purpose of this paper is to translate von Zeipel's proof into modern notation and terminology. In so doing, this author hopes to make von Zeipel's original ideas more readily available to current researchers and to help clarify von Zeipel's contribution to the theory of singularities in celestial mechanics.

2 Painlevé's Stockholm Lectures

In the autumn of 1895, at the invitation of Mittag-Leffler and under the sponsorship of Oscar II, King of Sweden and Norway, Paul Painlevé gave a series of lectures in Stockholm. These lectures were such an important event in the scientific community of Stockholm that the first lecture was attended by the king himself. Since von Zeipel's work on singularities was clearly inspired by that of Painlevé, we begin by describing Painlevé's theorems on the subject. Painlevé's

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lecture notes were published in 1897 [6] and were reproduced in his collected works [7].

Let $m_i > 0$ be the mass of particle i, and let $q_i \in R^3$ be its position. The potential energy of this system of n particles is given by

$$-U(q_1, \ldots, q_n) \equiv -\sum_{i<j} \frac{m_i m_j}{|q_i - q_j|},$$

where $|\cdot|$ denotes the Euclidean norm in R^3. The Newtonian formulation for the motion of this system of particles can be written

$$m_i \ddot{q}_i = \nabla_i U(q_1, \ldots, q_n), \quad i = 1, \ldots, n. \tag{1}$$

Here the symbol ∇_i denotes the gradient with respect to the i^{th} variable, while the double dot denotes the second derivative with respect to time t.

The potential energy fails to be defined whenever two or more of the particles coincide. If we write

$$q \equiv (q_1, \ldots, q_n) \in (R^3)^n,$$

then we can denote this singular set by

$$\Delta \equiv \bigcup_{i<j} \Delta_{ij}.$$

where

$$\Delta_{ij} \equiv \{q \in (R^3)^n \colon q_i = q_j\}. \tag{2}$$

Note that $U\colon (R^3)^n - \Delta \to (0, \infty)$ is real-analytic.

If we are given an initial position $q(0) \in (R^3)^n - \Delta$ and an initial velocity $\dot{q}(0) \in (R^3)^n$, then the standard existence and uniqueness theory for ordinary differential equations gives us the existence of a unique solution $q(t)$ defined for $t \in [0, t^*)$, where $t^* \in (0, \infty]$ is chosen to be maximal. The vector q is a real analytic function of t, and the point t^*, if it is finite, is a singularity of this function.

Definition. If $t^* < \infty$, then the solution $q(t)$ is said to experience a *singularity* at t^*.

In his Stockholm lectures Painlevé investigated the nature of the singularities which can occur. He proved the following theorem, which states that the minimum distance between all pairs of particles must approach zero at a singularity. A nice proof can be found also in the book of Siegel and Moser [9]. Here we have introduced function $\varrho\colon (R^3)^n \to [0, \infty)$ defined by

$$\varrho(q) \equiv \min_{i<j} |q_i - q_j|.$$

Theorem 1. *If $q(t)$ experiences a singularity at t^*; then $\varrho(q(t)) \to 0$ as $t \to t^*$.*

A slightly different interpretation of this theorem can be given if one notes that

$$\varrho(q) = \sqrt{2}\, d(q, \Delta),$$

where $d(q, \Delta)$ denotes the standard Euclidean distance in R^{3n} from the point q to the set Δ. Theorem 1 states that $q(t) \to \Delta$ as $t \to t^*$.

It is natural to ask whether $q(t)$ must approach some definite point on Δ. *A priori*, $q(t)$ might oscillate wildly while approaching Δ, or it might become unbounded as the distance to Δ goes to zero. If $q(t)$ does approach some point q^* as $t \to t^*$, then each of the particles has some limiting position at time t^*. Since $q^* \in \Delta$, at least two of these limiting positions must coincide, which means that these particles must collide as $t \to t^*$. Painlevé called such an event a "collision". A singularity which is not due to a collision he called a "pseudocollision".

Definition. Suppose that $q(t)$ has a singularity at t^*. This singularity is called a *collision* if there exists a $q^* \in \Delta$ such that $q(t) \to q^*$ as $t \to t^*$. Otherwise, the singularity is called a *pseudocollision*.

Painlevé wondered whether pseudocollisions can occur and gave credit, without a specific reference, to Poincaré for having suggested the concept [6, p. 588]. Painlevé did succeed in showing that pseudocollisions cannot occur for the three-body problem.

Theorem 2. *For $n=3$ all singularities are collisions.*

Painlevé also gave a sufficient condition for a singularity in the n-body problem to be a collision. However, the condition does little more than simply rule out behavior more complicated than simultaneous triple collisions and does not provide much further insight into the nature of these singularities. Painlevé ended his Stockholm lectures with the unresolved question of whether pseudocollisions can exist for $m \geq 4$. This question remains unresolved today, although there is strong evidence supporting the existence of pseudocollisions [2, 5].

3 Von Zeipel's Theorem

Edvard Hugo von Zeipel was born in Sweden in 1873, the grandson of a German immigrant. He was educated in Stockholm and Uppsala, receiving his Ph.D. from Uppsala University in 1904. His thesis involved a study of periodic orbits of the third kind in the three-body problem. He studied in Paris from June 1904 through September 1906, taking courses from Poincaré in celestial mechanics and from Painlevé in rational mechanics [4]. It is reasonable to suppose that von Zeipel's interest in singularities grew out of his association with Poincaré and Painlevé, since his paper on the subject appeared in May 1908, less than two years after he left Paris.

Here is von Zeipel's theorem as it appears in his paper. This author has taken the liberty of translating it from French into English.

Theorem. *If some of the particles do not tend to finite limiting positions as t approaches t_1, then one has necessarily*

$$\lim_{t \to t_1} R = \infty,$$

where R is the maximum of the mutual distances.

In view of the definitions given above, von Zeipel's theorem states that a singularity must be due to collision if the system of particles remains bounded. In more picturesque language we can say that the only way a pseudocollision can occur is for the system of particles to explode to infinity in finite time.

After 1908 von Zeipel seems to have drifted away from the mathematical aspects of celestial mechanics and into the more practical side of astronomy. He continued to work on the motions of comets and small planets, but he become more interested in the structure and evolution of stars. He was elected to the Swedish Royal Academy of Sciences in 1915 and was appointed to a personal chair of astronomy at Uppsala in 1919. He served as chairman of the Swedish Astronomical Society from 1926 to 1935 and as chairman of the National Committee for Astronomy from 1931 to 1948. In 1930 he won the Morrison prize from the New York Academy of Sciences for his work on the evolution and constitution of the stars. It is interesting to note that Mittag-Leffler chose von Ziepel to contribute an article to volume 38 of *Acta Mathematica*, which was published in 1920 and devoted to the work of Poincaré.

Despite von Zeipel's rather successful career, his theorem on singularities seems to have fallen into obscurity for a number of years. In 1920 Jean Chazy published a paper in *Comptes Rendus* announcing the same theorem [1]. He gave no reference to von Zeipel's work, so one must assume that he was completely unaware of it. Writing in 1941, Wintner was aware of von Zeipel's paper but was somewhat skeptical of it [14, p. 431]. By 1970, according to Sperling, "von Zeipel's statement seems to be virtually unknown" [10, p. 15].

Fortunately, interest in the subject of singularities in celestial mechanics was renewed in the early 1970's, due largely to the work of Pollard and Saari. In 1972 Saari extended von Zeipel's result to show that no pseudocollision can occur if the moment of inertia is "slowly varying" [8]. In 1974 Mather and McGehee constructed a solution of the four-body problem which becomes unbounded in finite time [5]. However, their solution contains an infinite number of double collisions which have been extended by an "elastic bounce" and hence is not an example of a pseudocollision. Gerver recently has given an indication of a construction of a pseudocollision for the five-body problem [2], but the details apparently are not yet complete.

Today Painlevé question of whether pseudocollisions exist remains unresolved, although there seems little doubt that the answer is yes. Even after this question is answered, von Zeipel's theorem will remain one of the fundamental contributions to the field.

4 A decomposition of the system

An important element in von Zeipel's proof is his decomposition of the moment of inertia of the system of particles into components corresponding to subsystems. Today this decomposition is best understood in terms of the geometry of the space $(R^3)^n$ with inner product

$$\langle q, p\rangle \equiv \sum_{i=1}^{n} m_i(q_i, p_i),$$

where $(\cdot, \cdot)$ denotes the standard inner product on R^3. The *moment of inertia* is defined to be the norm induced by this inner product, i.e.,

$$I(q) \equiv \|q\|^2 = \sum_{i=1}^{n} m_i|q_i|^2.$$

We denote the gradient of U with respect to this inner product by ∇U. That is, $\nabla U(q)$ is the vector in $(R^3)^n$ such that

$$\langle \nabla U(q), p\rangle = DU(q)p, \qquad \text{for all } p \in (R^3)^n,$$

where $DU(q): (R^3)^n \to R^1$ denotes the derivative of U. The equations of motion (1) then can be written

$$\ddot{q} = \nabla U(q). \tag{3}$$

Recall that the n particles are labeled with the integers 1 through n. Denote this set of labels by

$$N \equiv \{1, 2, \dots, n\}.$$

If μ is a subset of N, we can arbitrarily identify as a *subsystem* those particles whose labels are in μ. The set of points on the singular set Δ corresponding to coincidence between all the particles in the subsystem is

$$\Delta_\mu \equiv \{q \in (R^3)^n: q_i = q_j \text{ for all } i, j \in \mu\}.$$

Points in Δ_μ can be regarded as points of "total collapse" of the subsystem μ. If μ is empty, then Δ_μ is undefined. If μ contains a single point, then $\Delta_\mu = (R^3)^n$. If $\mu = \{i, j\}$, then $\Delta_\mu = \Delta_{ij}$, as defined in formula (2).

Now let ω be a partition of N, that is, a set of mutually disjoint subsets of N whose union is all of N. A partition of N corresponds to a decomposition of the total system into subsystems, each corresponding to one of the elements of the partition. The set of points corresponding to total collapse simultaneously in each subsystem is the linear subspace

$$\Delta_\omega \equiv \bigcap_{\mu \in \omega} \Delta_\mu.$$

The distinction between Δ_ω, where ω is a partition of N, and Δ_μ, where μ is a subset of N, should be clear from the context.

If μ is a subset of N, then the *center of mass* of the corresponding subsystem is defined as

$$c_\mu q \equiv (\sum_{i\in\mu} m_i q_i)/(\sum_{i\in\mu} m_i).$$

We use this physical quantity to define a linear map

$$\pi_\omega: (R^3)^n \to (R^3)^n: (\pi_\omega q)_i \equiv c_\mu q \quad \text{if } i\in\mu\in\omega.$$

It is easy to check that π_ω is an orthogonal projection with range Δ_ω and nullspace

$$X_\omega \equiv \{q\in(R^3)^n: \sum_{i\in\mu} m_i q_i = 0 \text{ for all } \mu\in\omega\}.$$

Thus, for each partition ω, $(R^3)^n$ can be written as the direct sum of the orthogonal subspaces Δ_ω and X_ω. If we write $\Pi_\omega \equiv \mathrm{id} - \pi_\omega$, then Π_ω denotes the orthogonal projection of $(R^3)^n$ onto X_ω. Thus we have

$$\|q\|^2 = \|\pi_\omega q\|^2 + \|\Pi_\omega q\|^2. \tag{4}$$

We digress briefly to give a physical interpretation of this last equation. We compute that

$$I_\omega(q) \equiv \|\pi_\omega q\|^2 = \sum_{\mu\in\omega} (\sum_{i\in\mu} m_i)|c_\mu q|^2.$$

Thus $I_\omega(q)$ is the moment of inertia of a system of particles consisting of, for each $\mu\in\omega$, a fictitious particle of mass $\sum_{i\in\mu} m_i$ located at the center of mass of the subsystem corresponding to μ. We also compute that

$$J_\omega(q) \equiv \|\Pi_\omega q\|^2 = \sum_{\mu\in\omega} J_\mu(q),$$

where

$$J_\mu(q) \equiv \sum_{i\in\mu} m_i |q_i - c_\mu q|^2.$$

Thus $J_\mu(q)$ is the moment of inertia with respect to its center of mass of the subsystem corresponding to μ. Equation (4) states that the total moment of inertia can be decomposed into the sum of the moments of inertia of each of the subsystems plus the moment of inertia of a system composed of a fictitious particle at the center of mass of each subsystem.

The potential energy can be decomposed in an analogous way. If we let

$$U_{ij}(q) \equiv \begin{cases} \dfrac{1}{2}\dfrac{m_i m_j}{|q_i - q_j|} & \text{for } i \neq j, \\ 0 & \text{for } i = j, \end{cases}$$

then we write

$$U(q)=\sum_{i\in N}\sum_{j\in N}U_{ij}(q).$$

When viewed as isolated from the rest of the system, the subsystem corresponding to μ has potential energy

$$V_\mu(q)\equiv\sum_{i\in\mu}\sum_{j\in\mu}U_{ij}(q).$$

For a partition ω we write

$$V_\omega(q)\equiv\sum_{\mu\in\omega}V_\mu(q), \tag{5}$$

which is the total potential energy of all the isolated subsystems. The remaining potential energy,

$$U_\omega(q)\equiv U(q)-V_\omega(q), \tag{6}$$

is due to the interactions between the subsystems. To be more precise, if we let

$$U_{\mu\nu}(q)\equiv\begin{cases}\sum_{i\in\mu}\sum_{j\in\nu}U_{ij}(q) & \text{for } \mu\cap\nu=\emptyset,\\ 0 & \text{for } \mu=\nu,\end{cases}$$

then we can write

$$U_\omega(q)=\sum_{\mu\in\omega}\sum_{\nu\in\omega}U_{\mu\nu}(q). \tag{7}$$

An examination of formula (5) yields the following identity:

$$V_\omega(q+z)=V_\omega(q) \quad \text{for all } z\in\Delta_\omega.$$

We therefore have that

$$V_\omega(q+\pi_\omega p)=V_\omega(q) \quad \text{for all } p\in(R^3)^n.$$

Differentiating with respect to p and setting $p=0$, we obtain

$$DV_\omega(q)\pi_\omega=0,$$

which, since π_ω is orthogonal, yields

$$\pi_\omega\nabla V_\omega(q)=0.$$

Combining this last equation with (3) and (6), we obtain

$$\pi_\omega\ddot{q}=\pi_\omega\nabla U_\omega(q), \tag{8}$$

using which we can compute that

$$\frac{d^2}{dt^2}I_\omega(q(t))=2\|\pi_\omega\dot{q}(t)\|^2+2\langle\pi_\omega q(t),\ \nabla U_\omega(q(t))\rangle. \tag{9}$$

5 The proof of von Zeipel's theorem

We conclude this note by translating von Zeipel's proof into the notation developed in the previous section. The proof begins with the following lemma, which states that the moment of inertia must approach a limit, possibly infinite, as $t \to t^*$.

Lemma. *If $q(t)$ experiences a singularity at t^*, then there exists an $I^* \in [0, \infty]$ such that $I(q(t)) \to I^*$ as $t \to t^*$.*

This lemma is proved using Theorem 1 and a formula due to Lagrange. A nice exposition of the details can be found in the book of Siegel and Moser [9, p. 26]. Although Wintner credits this lemma to Painlevé in his Stockholm lectures [14, p. 434], this author can find no mention of it there. It was proved by Sundman in 1906 [11, p. 8] for the case $n=3$ and by von Zeipel in 1908 [15] for arbitrary n. These references are the earliest known to this author.

We now use the lemma to state and prove an equivalent version of von Zeipel's theorem.

Theorem. *If $q(t)$ experiences a singularity at t^* and if $I(q(t)) \to I^* < \infty$ as $t \to t^*$, then there exists a $q^* \in \Delta$ such that $q(t) \to q^*$ as $t \to t^*$.*

Proof. Let

$$\Delta^* \equiv \bigcap_{t<t^*} cl(q((t, t^*))),$$

where cl denotes the topological closure in $(R^3)^n$. Since $q((t, t^*))$ is a nonempty bounded set, its closure is nonempty and compact. Since Δ^* is written as the nested intersection of nonempty compact sets, it is itself nonempty and compact. Note that Δ^* is a subset of Δ and that $I(q) = I^*$ for all $q \in \Delta^*$.

For each partition ω, define

$$\Delta^*_\omega \equiv \Delta^* \cap \Delta_\omega.$$

From among all the partitions ω such that $\Delta^*_\omega \neq \emptyset$, choose one with minimal cardinality. For the remainder of this proof, we let ω denote this fixed partition. Note that this choice assures us that all the denominators in formula (7) for $U_\omega(q)$ are nonzero for all q in Δ^*_ω and hence that $U_\omega(q)$ is defined on this set. Since Δ^*_ω is compact, there exist a neighborhood G of Δ^*_ω in $(R^3)^n$ and a finite M such that

$$\|\nabla U_\omega(q)\| \leq M \quad \text{and} \quad |\langle \pi_\omega q, \nabla U_\omega(q)\rangle| \leq M \qquad \text{for all } q \in G. \tag{10}$$

We introduce the variables z and x by defining

$$z \equiv \pi_\omega q \in \Delta_\omega \quad \text{and} \quad x \equiv \Pi_\omega q \in X_\omega.$$

We identify the Cartesian product of X_ω and Δ_ω with their direct sum, writing

$$(x, z) \equiv x + z = q \in (R^3)^n.$$

One of the following two cases must hold.

Case 1. Δ^* is not a subset of Δ_ω.

Case 2. Δ^* is a subset of Δ_ω.

First assume case 1. Choose an open subset B of Δ_ω, whose closure $\bar{B}$ is compact, such that $\Delta_\omega^* \subset B \subset \bar{B} \subset G$. Denote the boundary of B by $\partial B \equiv \bar{B} - B$. For each $\sigma > 0$ define

$$D_\sigma \equiv \{x \in X_\omega : \|x\| < \sigma\}.$$

Again let $\bar{D}_\sigma$ denote the closure and ∂D_σ denote the boundary of D_σ. Write

$$K_\sigma \equiv \bar{D}_\sigma \times \bar{B} \subset (R^3)^n.$$

Since ∂B is compact and since $\Delta^* \cap \partial B = \emptyset$, there exist a $\sigma_0 > 0$ and a $t_0 < t^*$ such that

$$q([t_0, t^*)) \cap (\bar{D}_{\sigma 0} \times \partial B) = \emptyset. \tag{11}$$

Assume that σ_0 is chosen small enough so that

$$K_{\sigma 0} \subset G. \tag{12}$$

Since Δ^* is not a subset of Δ_ω, there exists a $\sigma \in (0, \sigma_0)$ such that, for infinitely many values of t close to t^*, $q(t) \notin K_\sigma$. Henceforth we fix σ at this value. Choose t_1 so close to t^* that

$$|I(q(t)) - I^*| < \sigma^2/12 \qquad \text{for } t_1 \le t < t^*. \tag{13}$$

Since $q(t)$ comes infinitely often arbitrarily close to Δ_ω^* as $t \to t^*$, $q(t)$ must enter and leave K_σ infinitely often as $t \to t^*$. Property (11) implies that $q(t)$ must enter and leave via $\partial D_\sigma \times B$, so long as $t > t_1$. We therefore can find an interval $[\tau_0, \tau_3]$ satisfying the following conditions:

$$q(t) \in K_\sigma \qquad \text{for } \tau_0 \le t \le \tau_3, \tag{14}$$

$$J_\omega(q(\tau_0)) = J_\omega(q(\tau_3)) = \sigma^2, \tag{15}$$

$$\min_{\tau_0 \le \tau \le \tau_3} J_\omega(q(\tau)) < \sigma^2/2, \quad \text{and} \tag{16}$$

$$\tau_3 - \tau_0 < \sigma/\sqrt{3M}. \tag{17}$$

Condition (16) can be met since $q(t)$ comes arbitrarily close to Δ_ω^* for values of t arbitrarily close to t^*. Condition (17) can be met since intervals satisfying the first three conditions occur arbitrarily close to t^*.

Now let $\bar{\tau}$ be a point in (τ_0, τ_3) where $I_\omega(q(t))$ achieves its maximum value. Equation (4) implies that $I(q) = J_\omega(q) + I_\omega(q)$. Equations (13) and (16) imply that

$$I_\omega(q(\bar{\tau})) > I^* - 7\sigma^2/12,$$

while equations (13) and (15) imply that

$$I_\omega(q(\tau_3)) < I^* - 11\sigma^2/12.$$

Combining these last two inequalities, we find that

$$I_{\omega}(q(\bar{\tau}))-I_{\omega}(q(\tau_3))>\sigma^2/3. \tag{18}$$

On the other hand, (9), (10), (12), and (14) imply that

$$\frac{d^2}{dt^2}I_{\omega}(q(t))\geq -2M \qquad \text{for } \tau_0\leq t\leq \tau_3,$$

Since $\bar{\tau}$ is a local maximum, we therefore have that

$$I_{\omega}(q(\tau_3))-I_{\omega}(q(\bar{\tau}))\geq -M(\tau_3-\bar{\tau})^2.$$

Condition (17) now implies that

$$I_{\omega}(q(\bar{\tau}))-I_{\omega}(q(\tau_3))<\sigma^2/3,$$

which contradicts (18). Thus case 1 is impossible.

We have shown that case 2 must hold, i.e. that Δ^* must be a subset of Δ_{ω}. It follows immediately that $x(t)\to 0$ as $t\to t^*$. Furthermore, since $\Delta^*=\Delta^*_{\omega}$, G is a neighborhood of Δ^*. Therefore there exists a t_2 such that $q(t)\in G$ for $t_2<t<t^*$. Equation (8) implies that

$$\ddot{z}(t)=\pi_{\omega}\nabla U_{\omega}(q(t)),$$

which, when combined with (10), implies that

$$\|\ddot{z}(t)\|\leq M \qquad \text{for } t_2<t<t^*.$$

It follows that $z(t)$ must approach a limit $q^*\in\Delta_{\omega}$ as $t\to t^*$. Therefore

$$q(t)=x(t)+z(t)\to 0+q^* \qquad \text{as } t\to t^*,$$

and the proof is complete.

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