Solutions to Homework 11

FM 5011 Mathematical Background for Finance

Let W_t be a Brownian motion.

1-1. Compute $\mathbb{E}\left[W_1^2 W_2\right]$.
Let $Y := W_1$ and $Z := W_2 - W_1$. Since W_t is a Brownian motion, Y and Z are standard normal and iid. Since

$$Y^2 Z + Y^3 = W_1^2(W_2 - W_1) + W_1^3 = W_1^2 W_2,$$

$$\begin{aligned}
\mathbb{E}\left[W_1^2 W_2\right] &= \mathbb{E}\left[Y^2 Z + Y^3\right] \\
&= \mathbb{E}\left[Y^2 Z\right] + \mathbb{E}\left[Y^3\right] \\
&= \mathbb{E}\left[Y^2\right] \cdot \mathbb{E}\left[Z\right] + \mathbb{E}\left[Y^3\right] \qquad \text{(since } Y^2 \text{ and } Z \text{ are independent)} \\
&= 1 \cdot 0 + 0 \\
&= 0,
\end{aligned}$$

because

$$\begin{aligned}
\mathbb{E}\left[Y^2\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \left[-x e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right] = 0 + 1 = 1, \\
\mathbb{E}\left[Z\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \left[-e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} \right] = 0, \\
\mathbb{E}\left[Y^3\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \left[-x^2 e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 2x e^{-\frac{x^2}{2}} dx \right] = 0 + 0 = 0.
\end{aligned}$$

1-2. Let X_t be a solution to $dX_t = 3dW_t - 4dt$. Compute $\mathbb{E}\left[e^{X_2 + X_3}\right]$.
The solution to the SDE is

$$X_t = 3W_t - 4t.$$

Let $Y := \frac{W_2}{\sqrt{2}}$ and $Z := W_3 - W_2$. Since W_t is a Brownian motion, Y and Z are standard normal and iid.

$$\begin{aligned}
&Y = \frac{W_2}{\sqrt{2}} \implies W_2 = \sqrt{2} Y \\
&Z = W_3 - W_2 \implies W_3 = W_2 + Z = \sqrt{2} Y + Z.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[e^{X_2+X_3}\right] &= \mathbb{E}\left[e^{3W_2-8+3W_3-12}\right] \\
&= \mathbb{E}\left[e^{6\sqrt{2}Y+3Z-20}\right] \\
&= e^{-20}\mathbb{E}\left[e^{6\sqrt{2}Y+3Z}\right] \\
&= e^{-20}\mathbb{E}\left[e^{6\sqrt{2}Y}\right]\cdot\mathbb{E}\left[e^{3Z}\right] \qquad \text{(since } e^{6\sqrt{2}Y} \text{ and } e^{3Z} \text{ are independent)} \\
&= e^{-20}e^{36}e^{\frac{9}{2}} \\
&= e^{\frac{41}{2}},
\end{aligned}$$

since

$$\begin{aligned}
\mathbb{E}\left[e^{6\sqrt{2}Y}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{6\sqrt{2}}e^{-\frac{x^2}{2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{6\sqrt{2}(x+6\sqrt{2})}e^{-\frac{(x+6\sqrt{2})^2}{2}}\,dx \quad \text{(replacing } x \text{ by } x+6\sqrt{2}) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{6\sqrt{2}x+72-\frac{x^2}{2}-6\sqrt{2}x-36}dx \\
&= \frac{e^{36}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx \\
&= e^{36}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[e^{3Z}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{3x}e^{-\frac{x^2}{2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{3(x+3)}e^{-\frac{(x+3)^2}{2}}\,dx \quad \text{(replacing } x \text{ by } x+3) \\
&= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} e^{3x+9-\frac{x^2}{2}-3x-\frac{9}{2}}dx \\
&= \frac{e^{\frac{9}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx \\
&= e^{\frac{9}{2}}.
\end{aligned}$$

Let X_t be a solution to the SDE (stochastic differential equation)

$$dX_t = \left(\cos\left(X_t^2\right)\right)dW_t + \left(\sin(X_t)\right)dt.$$

2-1. Let $Y_t := X_t^5$. Find the SDE for Y_t.

Using Itô's lemma, we obtain

$$
\begin{aligned}
dY_t &= \left[\frac{\partial Y_t}{\partial t} + \frac{\partial Y_t}{\partial X_t}(\sin(X_t)) + \frac{1}{2}\frac{\partial^2 Y_t}{\partial X_t^2}\left(\cos(X_t^2)\right)^2\right] dt + \frac{\partial Y_t}{\partial X_t}\left(\cos(X_t^2)\right) dW_t \\
&= \left[0 + 5X_t^4 \sin(X_t) + \frac{1}{2}\cdot 20X_t^3 \cos^2\left(X_t^2\right)\right] dt + 5X_t^4\cos(X_t^2)dW_t \\
&= \left[5X_t^4 \sin(X_t) + 10X_t^3\cos^2(X_t^2)\right] dt + 5X_t^4\cos(X_t^2)dW_t.
\end{aligned}
$$

Therefore, Y_t satisfies the stochastic differential equation

$$dY_t = 5X_t^4\cos(X_t^2)dW_t + \left[5X_t^4\sin(X_t) + 10X_t^3\cos^2(X_t^2)\right] dt.$$

2-2. Let $S_t := te^{X_t}$. Find the SDE for S_t.

Using Itô's lemma, we obtain

$$
\begin{aligned}
dS_t &= \left[\frac{\partial S_t}{\partial t} + \frac{\partial S_t}{\partial X_t}(\sin(X_t)) + \frac{1}{2}\frac{\partial^2 S_t}{\partial X_t^2}\left(\cos(X_t^2)\right)^2\right] dt + \frac{\partial S_t}{\partial X_t}\left(\cos(X_t^2)\right) dW_t \\
&= \left[e^{X_t} + te^{X_t}\sin(X_t) + \frac{1}{2}te^{X_t}\cos^2\left(X_t^2\right)\right] dt + te^{X_t}\cos(X_t^2)dW_t.
\end{aligned}
$$

Therefore, S_t satisfies the stochastic differential equation

$$dS_t = te^{X_t}\cos(X_t^2)dW_t + \left[e^{X_t} + te^{X_t}\sin(X_t) + \frac{1}{2}te^{X_t}\cos^2\left(X_t^2\right)\right] dt.$$