MATH 2373.020 – Lecture 23 – 4/15/09

HW12, due Tuesday, April 21:
7711 #1-4
7713 #1-4
7715 #1
7729 #1-8
7768 #1-5
7772 #1-5
7782 #1,2

Today:

- diagonalizing matrices
- finally, using $\mathcal{L}$ to solve DE
- two-tank problems, leading to 2x2 non-homogeneous systems

Diagonalizing matrices

A square matrix D is a *diagonal* matrix if all the entries off the main diagonal are zero.

A square matrix P is *singular* if its determinant is 0 and *non-singular* if its determinant is not 0.

A square matrix B is *diagonalizable* if there is a non-singular matrix P such that

$$P^{-1}BP = D.$$

Why, in our context, is it nice to work with diagonal matrices?

Diagonalizing the coefficient matrix of a system of DE will simplify its solution. This method is called "uncoupling".

Example 1. Diagonalize

$$B = \begin{bmatrix} 5 & 2 \\ -1 & 2 \end{bmatrix};$$

that is, find a non-singular matrix P such that $P^{-1}BP$ is a diagonal matrix.

- find the eigenvalues of B
- construct P from eigenvectors
- find BP
- find P^{-1} (remember: swap diagonal entries, change sign of off-diagonal entries, divide by det)
- $P^{-1}BP$ has eigenvalues of B down the diagonal

Notes:

1. Since eigenvectors are not unique, neither is the matrix P.

2. Why does P have that form?

3. Not every square matrix is diagonalizable.

4. When diagonalizing 3x3 matrices, for example, recall the procedure for finding P^{-1}: the transpose of the matrix of cofactors, divided by the determinant.

5. Another application of diagonalization: large powers of matrices.

Using Laplace transforms to solve differential equations

Example 2. Solve the initial value problem

$$y' + 2y = \cos t,$$

$$y(0) = 3.$$

Example 3. For your practice, from Monday:

Find

$$\mathcal{L}^{-1}\left\{\frac{6s^2+50}{(s+3)(s^2+4)}\right\}.$$

Solve the initial value problem

$$y' + 3y = 13\sin 2t,$$

$$y(0) = 6.$$

Example 4. Solve, using Laplace transforms, the initial value problem

$$y'' + 6y' + 8y = 0,$$

$$y(0) = 1,$$

$$y'(0) = 2.$$

Summary: How $\mathcal{L}$ treats the first and second derivative.

If $\mathcal{L}\{y(t)\} = Y(s)$, then

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

and

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0).$$

Example 5. Write a system of differential equations that models the quantities of salt in the two tanks in the following scenario:

Initially, tank A contains 5 lb of salt dissolved in 100 gal of water, and tank B contains 10 lb of salt dissolved in 150 gal of water.

Brine is flowing as follows:
1. into tank A, 3 gal/min, 2 lb/gal
2. into tank B, 4 gal/min, 3 lb/gal
3. from A to B, 1 gal/min
4. from B to A, 2 gal/min
5. draining from A, 4 gal/min
6. draining from B, 3 gal/min

The table:

$\mathcal{L}\{1\} = \frac{1}{s}$
$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$
$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$