Math 8669: Combinatorial Theory

HW 2 (Due Wednesday March 7, 2018)

Professor: Gregg Musiker

Remark: Please do **at least seven** of the following thirteen problems.

1) Exercises mentioned in Lectures 12-13:

(a) Consider the representation $\rho : S_4 \to GL_2(\mathbb{C})$ given by

$$\rho\left((12)\right) = \rho\left((34)\right) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$\rho\left((13)\right) = \rho\left((24)\right) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$\rho\left((14)\right) = \rho\left((23)\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Extending by multiplication (i.e. linearly), what do (123), (1324), (12)(34), and (132) map to under ρ?

(b) Letting $g = (1234)$, $h = (132)$, verify that

$$\rho(g)^4 = \rho(h)^3 = \rho(ghgh) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b') Alternatively: verify the braid relations for S_4, i.e.

$$\rho((12))^2 = \rho((23))^2 = \rho((34))^2 = \rho((12)(34))^2 = \rho((12)(23))^3 = \rho((23)(34)))^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) Verify that the symmetric function (pattern repeats with an infinite number of variables)

$$x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + 2x_1x_2x_3x_4 + \dots$$

decomposes as

$$\frac{p_1^4}{12} - \frac{p_3p_1}{3} + \frac{p_2p_2}{4} = \frac{1}{24}\left(1\cdot 2\cdot p_1^4 + 6\cdot 0\cdot p_2p_1p_1 + 8\cdot(-1)\cdot p_3p_1 + 6\cdot 0\cdot p_4 + 3\cdot 2\cdot p_2p_2\right).$$

2) (a) (Exercise 1 in Section 1.13 of Sagan) An *inversion* in a permutation $\pi = x_1, x_2, \dots, x_n \in S_n$ (in one-line notation) is a pair x_i, x_j such that $i < j$ and $x_i > x_j$.

Show that if π can be written as a product of k transpositions, then $k \equiv inv(\pi) \mod 2$. Note that this shows that the sign of π, defined as $sgn(\pi) = (-1)^k$, is well-defined.

(b) Exercise 7.3 in EC 2.

3) (Sagan, Problem 1.13.4) Let G be an abelian group. Find all inequivalent irreducible representations of G. *H*int: Use the fundamental theorem of abelian groups.

4) (Sagan, Problem 1.13.10) Verify that the map $X : \mathbb{R}_{>0} \to GL_2$ by $X(r) = \begin{bmatrix} 1 & \log r \\ 0 & 1 \end{bmatrix}$ is a representation and that $W = \{\begin{bmatrix} c \\ 0 \end{bmatrix} : c \in \mathbb{C}\}$ is invariant.

5) (Sagan, Problem 1.13.12) Let X be an irreducible matrix representation of G. Show that if $g \in Z_G$ (the center of G), then $X(g) = cI$ for some scalar c.

6) (Sagan, Problem 1.13.13) Let $\{X_1, X_2, \dots, X_n\} \subseteq GL_d$ be a subgroup of commuting matrices. Show that these matrices are simultaneously diagonalizable using representation theory.

7) (Sagan, Problem 1.13.15) Let X and Y be representations of G. The *inner tensor product*, $X\hat{\otimes}Y$, assigns to each $g \in G$ the matrix $(X\hat{\otimes}Y)(g) = X(g) \otimes Y(g)$.

   a) Verify that $X\hat{\otimes}Y$ is a representation of G.

   b) Show that if X, Y, and $X\hat{\otimes}Y$ have characters denoted as χ, ϕ, and $\chi\hat{\otimes}\phi$, respectively, then $(\chi\hat{\otimes}\phi)(g) = \chi(g)\phi(g)$.

   c) Find a group with irreducible representations X and Y such that $X\hat{\otimes}Y$ is not irreducible.

   d) However, prove that if X is of degree 1 and Y is irreducible, then so is $X\hat{\otimes}Y$.

8) (Sagan, Problem 1.13.17) Let D_n be the Dihedral group of symmetries (rotations and reflections) of a regular n-gon.

   c) Find the conjugacy classes of D_n.

   d) Find all the inequivalent irreducible representations of D_n. *H*int: Use the fact that C_n is a normal subgroup of D_n.

9) (Sagan, Problem 1.13.16) Construct the character table of S_4.

10) (Sagan, Problem 2.12.4) Consider $S^{(n-1,1)}$, where each tabloid is identified with the element in its second row. Prove the following facts about this module and its character.

    a) We have $S^{(n-1,1)} = \{c_1\mathbf{1} + c_2\mathbf{2} + \cdots + c_n\mathbf{n} : c_1 + c_2 + \cdots + c_n = 0\}$.

    b) For any $\pi \in S_n$, $\chi^{(n-1,1)}(\pi) =$ (the number of fixed points of π) -1.

11) (Sagan, Problem 2.12.6) Show that every irreducible character of S_n is an integer-valued function.

12) (Sagan, Problem 2.12.15-17) Compute the character table of the alternating group A_4. *H*int: You might find it useful to consider the character table of S_4 and how the conjugacy classes of A_n and S_n compare.

13) Use Frobenius Reciprocity to verify the following identities involving induced representations:

(a) $(trivial)_{\mathbb{Z}_2}^{S_3} = (two - dim\ irrep) \oplus (trivial)$,

(b) $(sign)_{\mathbb{Z}_2}^{S_3} = (two - dim\ irrep) \oplus (sign)$,

(c) $(trivial)_{\mathbb{Z}_3}^{S_3} = (trivial) \oplus (sign)$,

(d) $(\omega)_{\mathbb{Z}_3}^{S_3} = (\omega)^{2}{}_{\mathbb{Z}_3}^{S_3} = (two - dim\ irrep)$. where ω is the representation that sends the generator of $\mathbb{Z}_3$ to a primitive cube root of unity.