Math 8680: Cluster Algebras and Quiver Representations

Homework 1 (Due Wednesday February 16, 2011)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators. You may use computer algebra packages for calculations but should also briefly describe your calculations in words in this case.

1) Consider the cluster algebra of geometric type defined by the initial labeled seed given by

$$\mathbf{x} = \{x_1, x_2, u_1, u_2, u_3\} \text{ and } B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

a) Compute all cluster variables generating this cluster algebra.

b) Try to find a geometric model for this cluster algebra.

2) In this problem, you will describe the connection between the original definition of cluster algebras and those of geometric type.

Let $\mathbb{P}$ be the semifield $\mathrm{Trop}(u_1, u_2, \ldots, u_m)$ defined with $\oplus$ as $u_1^{d_1} \cdots u_m^{d_m} \oplus u_1^{e_1} \cdots u_m^{e_m} = u_1^{\min(d_1,e_1)} \cdots u_m^{\min(d_m,e_m)}$. Let $\mathcal{A}$ be a rank n cluster algebra defined over the ground ring $\mathbb{ZP}$ with seed

$$(\mathbf{x} = \{x_1, x_2, \ldots, x_n\}, \quad \mathbf{y} = \{y_1, y_2, \ldots, y_n\}, \quad B).$$

We remind the reader that the set $\mathbf{x}$ is algebraically independent, the y_i's are in $\mathbb{P}$, and B is an n-by-n skew-symmetrizable matrix. Elements of $\mathbb{ZP}$ are Laurent polynomials in the u_i's.

Let $\tilde{\mathcal{A}}$ denote a cluster algebra of *geometric type* (over the ground ring $\mathbb{Z}[u_1^{\pm 1}, \ldots, u_m^{\pm 1}]$) with seed given by

$$\left(\tilde{\mathbf{x}} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, u_1, u_2, \ldots, u_m\}, \quad \tilde{B}\right),$$

where $\tilde{B}$ is an $(m+n)$-by-n matrix, whose top n-by-n submatrix is skew-symmetrizable.

a) Show that if $\tilde{B} = \left[\frac{B}{C}\right]$ where $C = [c_{ij}]$ is a general m-by-n integer matrix such, and we let $x_i = \tilde{x}_i$, $y_j = \prod_{i=1}^{m} u_i^{c_{ij}}$ for each column index $j \in \{1, 2, \ldots, n\}$, then $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are isomorphic as cluster algebras.

In particular, show that the mutation rules (1) and (2) agree, and that the matrix mutation rule for the matrix $\tilde{B}$ is equivalent to the matrix mutation rule for B plus the coefficient mutation rule (3).

$$x'_k = \frac{y_k \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{(y_k \oplus 1) x_k} \tag{1}$$

$$\tilde{x}'_k = \frac{\prod_{\substack{\tilde{b}_{n+i,k}>0 \\ 0\le i\le m}} u_i^{\tilde{b}_{n+i,k}} \prod_{\substack{\tilde{b}_{ik}>0 \\ 0\le i\le n}} \tilde{x}_i^{\tilde{b}_{ik}} + \prod_{\substack{\tilde{b}_{n+i,k}<0 \\ 0\le i\le m}} u_i^{-\tilde{b}_{n+i,k}} \prod_{\substack{\tilde{b}_{n+i,k}<0 \\ 0\le i\le n}} \tilde{x}_i^{-\tilde{b}_{ik}}}{\tilde{x}_k} \tag{2}$$

$$y'_j = \begin{cases} y_k^{-1} & \text{if } \ j = k, \text{ and} \\ y_j \ \frac{y_k^{\max(b_{kj},0)}}{(y_k \oplus 1)^{b_{kj}}} & \text{if } \ 1 \le j \le n, j \ne k \end{cases} \tag{3}$$

b) As an example, reformulate the cluster algebra of geometric type described in Problem 1 as a cluster algebra with coefficients y_1, $y_2 \in \text{Trop}(u_1, u_2, u_3)$. Compute $\mu_1(y_1), \mu_1(y_2), \mu_2\mu_1(y_1)$, and $\mu_2\mu_1(y_2)$.

3) a) Let $\{f_n\}$ be a sequence of rational functions defined by the following initial conditions and recurrence: $f_1 = x_1,\ f_2 = x_2,\ f_3 = x_3$, and for $n \ge 3$, $f_n f_{n-3} = f_{n-1} f_{n-2} + 1$. Use the Caterpillar Lemma or model this sequence by a cluster algebra to show that f_n is a Laurent polynomial in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ for all $n \ge 1$.

b) By a similar method, show that Somos-5, defined as $f_1 = x_1,\ f_2 = x_2,\ f_3 = x_3,\ f_4 = x_4,\ f_5 = x_5$, and for $n \ge 5$, $f_n f_{n-5} = f_{n-1} f_{n-4} + f_{n-2} f_{n-3}$ gives rise to Laurent polynomials in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}, x_5^{\pm 1}]$.

4) In this problem, you will prove some of the results needed for the *finite type classification*.

a) Prove that an n-by-n matrix B is skew-symmetrizable if and only if following two conditions on B hold:

(i) For all pairs $1 \le i, j \le n$, either b_{ij} and b_{ji} are both zero, or the product $b_{ij} b_{ji}$ is negative. This condition is called *sign-skew-symmetric*.

(ii) For all subsequences $[i_1, i_2, \ldots, i_k]$ of $\{1, 2, \ldots, n\}$, we have

$$b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_k i_1} = (-1)^k b_{i_2 i_1} b_{i_3 i_2} \cdots b_{i_1 i_k}.$$

We say that a skew-symmetrizable matrix B is 2-*finite* if B satisfies the bound $|b'_{ij} b'_{ji}| \le 3$ for all i, j and all $B' = [b_{ij}]$ mutation-equivalent to B. Note that B is 2-finite if and only if $\mu_k(B)$ is 2-finite (for any k).

b) Prove that if B is not 2-finite, then the corresponding cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is not of finite type.

c) Prove that if B is skew-symmetrizable and 2-finite, then every triangle in the diagram $\Gamma(B)$ of B is (i) oriented cyclically, and (ii) has weights $\{1, 1, 1\}$ or $\{1, 2, 2\}$.

d) Using (c) or the definition above, show that the following matrices are not 2-finite:

$$A_2^{(1)} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix},\ B_3^{(1)} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 1 & -1 & 1 & 0 \end{bmatrix},\ C_3^{(1)} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ \text{and } D_5^{(1)} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$