Math 8680: Cluster Algebras and Quiver Representations

Homework 2 (Due Wednesday March 30, 2011)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators. You may use computer algebra packages for calculations but should also briefly describe your calculations in words in this case.

1) Let Q_n be the unidirectional quiver of type A_n which consists of n vertices linearly ordered and labeled as $\{1, 2, \ldots, n\}$, and arrows a_i between vertex $\mathbf{i}$ and $(\mathbf{i+1})$. Let k be a field and kQ_n be the associated path algebra. Let $B_n(k)$ denote the k-algebra of lower-triangular n-by-n matrices over field k.

   a) Show that kQ_n and $B_n(k)$ are isomorphic as k-algebras.

   Hint: It suffices to exhibit a map between basis elements and then show that the same relations are satisfied in both k-algebras.

   b) Describe the projective and injective indecomposable representations of Q_n.

2) (Exercise 2 of Derksen Lecture 5) Suppose that A is a positive-definite symmetric $n \times n$ matrix with $A_{i,i} = 2$ for all i and $A_{i,j} \in \{-1, 0\}$ for $i \neq j$. To this matrix we can draw a graph by drawing an edge between i and j ($i \neq j$) if and only if $A_{i,j} = -1$.

   Hint: You may use Sylvester's Criterion: A real symmetric matrix is positive-definite if and only if all of its upper-left principal minors have positive determinant.

   a) Show that the graph cannot have a loop. This graph also cannot have a larger cycle, but you do not necessarily need to show this.

   b) Show that the graph cannot have a vertex with ≥ 4 edges, or two distinct vertices with each ≥ 3 edges.

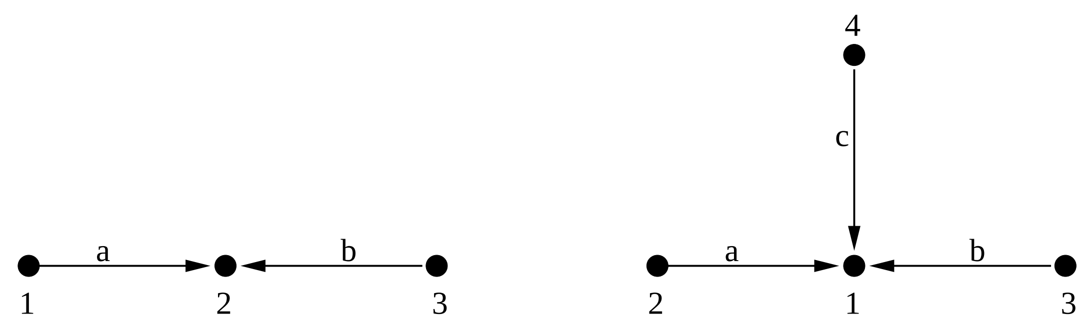
   c) The only possibility left for the graph is A_n or $T_{p,q,r}$ where $T_{p,q,r}$ is the graph with $p+q+r-2$ vertices which has exactly one vertex with 3 edges and this vertex has 3 strings of length p, q, r respectively.

   Suppose that the graph is $T_{p,q,r}$ for some p, q, r. Let $M_{p,q,r}$ be the $(p+q+r-2) \times (p+q+r-2)$ matrix corresponding to this graph (again 2's on the diagonal and 0 or -1 off the diagonal). Show that for $(p, q, r) = (2, 3, 6), (2, 4, 4), (3, 3, 3)$, the matrix $M_{p,q,r}$ is not positive definite (in fact, it is singular in these cases).

d) Conclude from above that for A to be positive definite and symmetric with 2's on the diagonal and only 0's or -1's off the diagonal, the corresponding graph must be A_n, D_n, E_6, E_7, or E_8.

Hint: Can also see Problem 5.3 of Etingof's notes `http://math.mit.edu/~etingof/replect.pdf`.

3) Construct all the indecomposable quiver representations (up to isomorphism) for (a) Q_{A_3} and (b) Q_{D_4} which are displayed from left to right in the graphic below. Note that we have a bipartite orientation, rather than a unidirectional orientation for A_3 in this case. Also identify which of these representations are projective, injective, or simple (at least for the Q_{A_3} case).



4) a) Let Q_{A_3} be the same quiver as in problem 3. For each indecomposable representation V of Q_{A_3}, compute the corresponding unique non-initial cluster variable X_V according to the Caldero-Chapoton formula.

Again using Q_{A_3}, let $\{i_1, i_2, i_3\}$ be an ordering of $\{1, 2, 3\}$ so that if we apply the BGP reflection functor at i_1, then at i_2, followed by i_3, then we only end up applying the functor $C^+_{i_j}$ at vertices that are sinks. (**Hint:** There are two such orderings for this quiver.) Let C^+ be defined as the composition $C^+_{i_3}C^+_{i_2}C^+_{i_1}$.

b) Show that C^+ is well-defined, i.e. it has the same image even no matter which of the two orderings we choose.

We define C^- analogously, acting only on sources.

c) Apply C^-C^+ repeatedly, starting with the projective indecomposables of Q_{A_3}.

d) Describe how the associated indecomposable representations (and the corresponding Laurent polynomials given by the Caldero-Chapoton formula) compare with the list of cluster variables obtained by mutating along the bipartite belt of the cluster algebra of type A_3.

Bonus: More difficult, but a useful exercise to do parts (a)-(d) for Q_{D_4}.