

Lecture 10: Rank Two Positive Bases

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Note Title

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① Recall that we let $A(b,c)$ denote a cluster algebra of rank two with initial exchange matrix $\begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$.

$A(b,c) \subset \mathbb{Q}(x_1, x_2)$ & has generators $\{x_n : n \in \mathbb{Z}\}$ satisfying

$$x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 & \text{if } n \text{ is even} \\ x_{n-1}^c + 1 & \text{if } n \text{ is odd.} \end{cases}$$

As shown last time, for any choice of initial cluster $\{x_m, x_{m+1}\}$, $m \in \mathbb{Z}$, and any choice of $n \in \mathbb{Z}$, then we can write

x_n as a Laurent polynomial

$$\sum_{\alpha} c_\alpha x_m^{\alpha_1} x_{m+1}^{\alpha_2} = \frac{p_n(x_m, x_{m+1})}{d_m(n) d_{m+1}(n)} x_m^{\alpha_1} x_{m+1}^{\alpha_2}$$

where p_n is a polynomial not divisible by x_m or x_{m+1} , and for $n \neq m, m+1$, $d_m(n), d_{m+1}(n) \geq 0$. $x_m = \frac{1}{x_{m+1}} \rightarrow x_{m+1} = \frac{1}{x_m}$

Def: A nonzero element $y \in A(b,c)$ is positive if for all $n \in \mathbb{Z}$, the coefficients in the expansion of y as a Laurent polynomial in $\{x_m, x_{m+1}\}$ are all positive.

(2) Note that this notion of "positive" is quite strong.

For example, if $(b, c) = (z, z)$ and

we let $y = x_0x_1 + x_1x_2 + x_3x_4 - x_0x_3$, then in

$$\{x_1, x_2\}, y = \left(\frac{x_1^2 + 1}{x_2}\right)x_1 + x_1x_2 +$$

$$\left(\frac{x_2^2 + 1}{x_1}\right)\left(\frac{(x_2^2 + 1)^2 + x_1^2}{x_1^2 x_2}\right) - \left(\frac{x_1^2 + 1}{x_2}\right)\left(\frac{x_2^2 + 1}{x_1}\right)$$

$$= \underbrace{(x_2^2 + 1)^3 + x_1^6}_{x_1^3 x_2} \text{ which is positive.}$$

However expanding the above in terms of cluster $\{x_0, x_1\}$, we get

$$y = x_0x_1 + x_1\left(\frac{x_1^2 + 1}{x_0}\right) + \left(\frac{(x_1^2 + 1)^2 + x_0^2}{x_0^2 x_1}\right)\left(\frac{(x_1^3 + 1)^3 + 2x_0^2(x_1^2 + 1)}{x_0^3 x_1^2 + x_0^4}\right) - x_0\left(\frac{(x_1^2 + 1)^2 + x_0^2}{x_0^2 x_1}\right) =$$

$$\left[(x_1^2 + 1)^5 + 3x_0^2(x_1^2 + 1)^3 + 3x_0^4(x_1^2 + 1) + x_0^6(x_1^4 + 1) - x_0^6 x_1^2 \right] / x_0^5 x_1^3$$

On the other hand, $z = x_0x_3 - x_1x_2$

expands as $\frac{x_m^2 + x_{m+1}^2 + 1}{x_m x_{m+1}}$ in any cluster $\{x_m, x_{m+1}\}$

③ and thus satisfies the definition of positivity.

The set of positive elements in $A(b,c)$ is a semi-ring, i.e. closed under addition and multiplication.

Thm [Sherman-Zelenitsky 2004]

Suppose that $bc \leq 4$, i.e. $A(b,c)$ is of finite ($bc \leq 3$) type or affine ($bc = 4$) type. Then there exists a unique \mathbb{K} -basis B of $A(b,c)$ s.t. the semi-ring of positive elements consists precisely of positive integer linear combinations of elements of B .

Remark: Since B is a basis its elements are linearly independent and each element of the semi-ring has a unique expression as an \mathbb{K} -linear combo of elements in B .

Further, uniqueness of B (assuming such a basis exists) is clear since B consists of the indecomposable positive elements (that cannot be written as a positive sum of other elements).

B is known as the (positive) canonical basis of $A(b,c)$.

In the rest of today's lecture, we explicitly construct B for $bc \leq 4$ and sketch the proof that it has the desired properties.

④ First some terminology: For $p, q \in \mathbb{Z}_{\geq 0}$
 we call $x_m^p x_{m+1}^q$ a cluster monomial.

This is because $\{x_m, x_{m+1}\}$ is a cluster of $A(b, c)$.

In general, if $\{x_{B_1}, \dots, x_{B_n}\}$ is a cluster of A , then $x_{B_1}^{c_1} \dots x_{B_n}^{c_n}$ is a cluster monomial.

Thm 1: If $bc \leq 3$, then the pos. canonical basis B is the set of all cluster monomials.

In the case that $bc = 4$, the affine

case, we define $\mathcal{Z} = \begin{cases} x_0 x_3 - x_1 x_2 & \text{if } (b, c) = (2, 2) \\ x_0^2 x_3 - (x_1 + 2)x_2^2 & \text{if } (b, c) = (1, 4) \end{cases}$

Let T_0, T_1, T_2, \dots be the sequence of (normalized)

Chebyshev polynomials of the 1st kind satisfying $\overline{T_0} = \mathcal{Z}$

$$T_n(t + t^{-1}) = t^n + t^{-n}$$

for $n > 0$. In particular, the T_K 's also satisfy

$$T_1(x) = x \text{ and } T_K = T_1 T_{K-1} - T_{K-2} \text{ for } K \geq 2.$$

$$[t^k + t^{-k}] = [t + t^{-1}] [t^k + t^{-k+1}] - [t^{k-2} + t^{-k+2}]$$

$T_0 = \mathcal{Z}$ differs from the $S\mathcal{Z}$ convention of $T_0 = 1$, but T_0 not used anyway in $[S\mathcal{Z}]$.

⑤ For $n > 0$, define $z_n := T_n(z)$.

Thm 2 If $(b, c) = (z, z)$ or $(1, 4)$,

then the pos. canonical basis β is
 $\{ \text{cluster monomials} \} \cup \{ z_n : n > 1 \}$.

E.g. by Thm 2, the element y ,
defined as $y = x_0 x_1 + x_3 x_4 - \frac{z}{11}$

could not be a positive element as it is not $x_0 x_3 - x_1 x_2$
a $\mathbb{Z}_{\geq 0}$ -linear combination of β elts.

To work towards the proofs of Thm 1 and Thm 2, we recall

the bijection between cluster vars and almost positive real roots.

$$\text{Each } X_n = \frac{P_n(x_1, x_2)}{x_1^{d_1(n)} x_2^{d_2(n)}} \leftrightarrow d_1(n)\alpha_1 + d_2(n)\alpha_2 \in \mathbb{Q} \cong \mathbb{Z}^2$$

We showed already that in the rank two $A(b, c)$ case, the

$d_1(n)\alpha_1 + d_2(n)\alpha_2$'s corresponding to non-initial cluster variables are the real positive roots in the image of $\begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}$ acting on $-\alpha_1, -\alpha_2$.

⑥ The set of positive roots can be enlarged to also contain what are known as imaginary roots.

In the rank two case, these are the roots roots $\underline{\alpha} = a_1\alpha_1 + a_2\alpha_2$ satisfying

$$\langle \underline{\alpha}, \underline{\alpha} \rangle = c a_1^2 - b c a_1 a_2 + b a_2^2 \leq 0.$$

In particular, it can be shown that nonzero real roots satisfy $\langle \underline{\alpha}, \underline{\alpha} \rangle > 0$, and this holds for general rank.

$$\underline{\Phi}_+ = \underline{\Phi}_+^{\text{Re}} \cup \underline{\Phi}_+^{\text{Im}} \left(\underline{\Phi} = \underline{\Phi}_+^{\text{Im}} \cup \underline{\Phi}_-^{\text{Im}} \right)$$

||
- $\underline{\Phi}_+$

Prop [SZ] Every cluster monomial of a rank two cluster algebra has a unique presentation of the form

$$X[\underline{\alpha}] = \frac{P_{\underline{\alpha}}(x_1, x_2)}{x_1^{a_1} x_2^{a_2}} \quad \text{where } P_{\underline{\alpha}} \text{ is a polynomial with constant term 1 and}$$

$$\underline{\alpha} = a_1\alpha_1 + a_2\alpha_2 \in Q - \underline{\Phi}_+^{\text{Im}}.$$

This gives a bijection between the set of cluster monomials and $Q - \underline{\Phi}_+^{\text{Im}}$. (For any $b_j c \geq 0$)

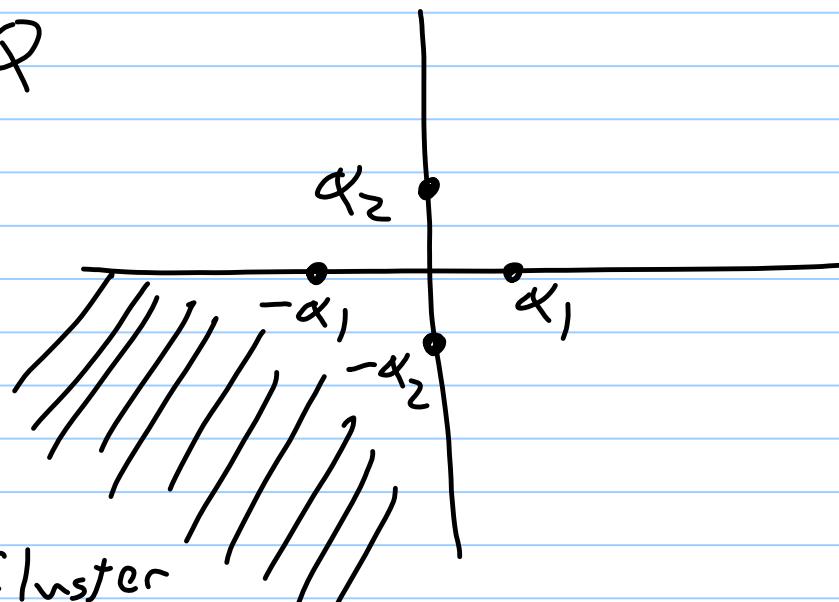
Remark: This bijection generalizes the one between cluster variables and almost positive real roots.

E.g.
 ⑦ $x_1^p x_2^q$ is a cluster monomial
 for any choice of $p, q \geq 0$

Since $x_1 \longleftrightarrow -\alpha_1, x_2 \longleftrightarrow -\alpha_2$

$$x_1^p x_2^q \longleftrightarrow -p\alpha_1 - q\alpha_2$$

Q



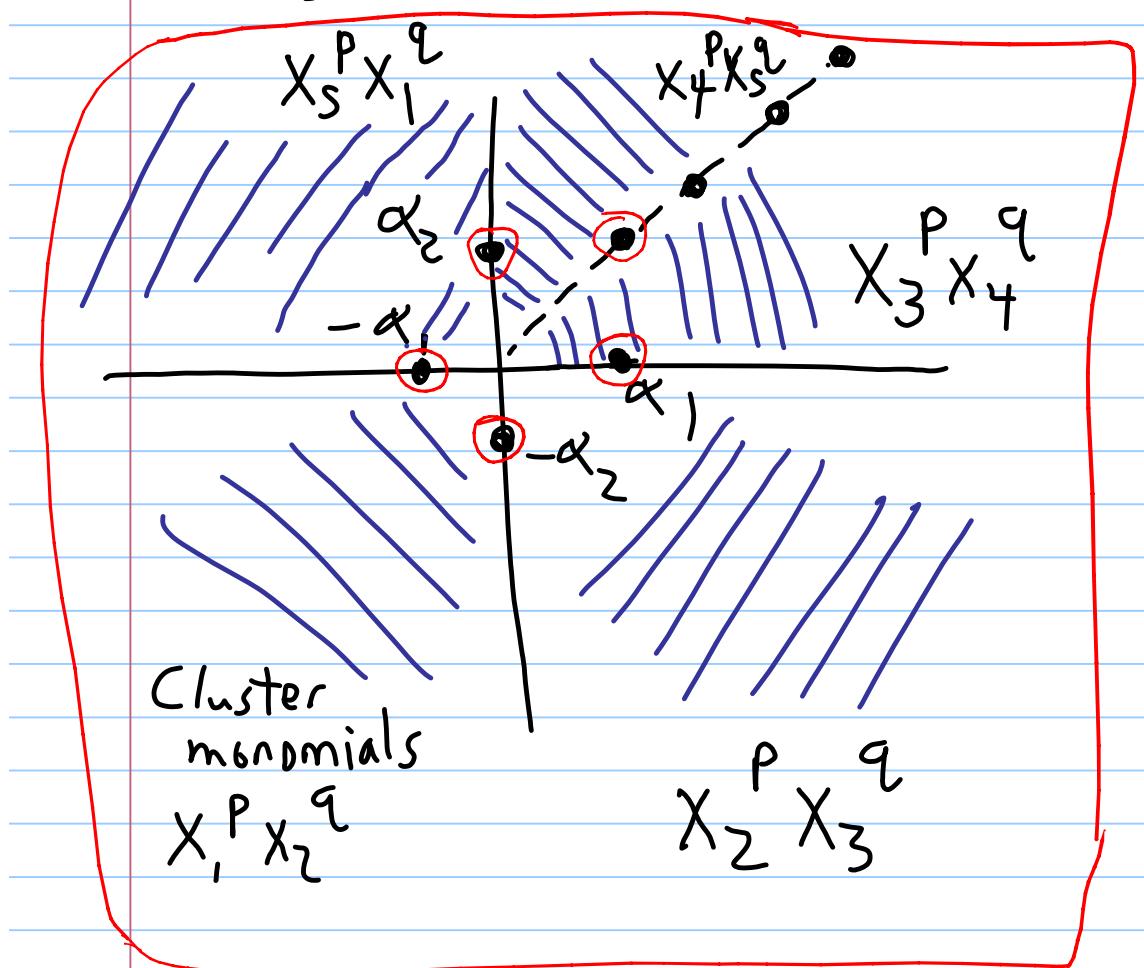
If in the case $(b_j, c) = (1, 1)$, i.e.
 A_2 , then the other possible

$$\text{clusters are } \left\{ x_2, x_3 = \frac{x_2+1}{x_1} \right\},$$

$$\left\{ x_3, x_4 = \frac{x_1+x_2+1}{x_1 x_2} \right\}, \left\{ x_4, x_5 = \frac{x_1+1}{x_2} \right\},$$

and $\{x_5, x_1\}$. There are no
 imaginary roots in this case as
 $\langle \alpha, \alpha \rangle = (\alpha, -\alpha)^2 + \alpha_1 \alpha_2 > 0$ unless $\alpha_1 = \alpha_2 = 0$.

⑧ Thus, we can fill out $Q \cong \mathbb{Z}^2$ us



Notice in A_2 case, every cluster monomial corresponds to a unique elt of $Q \cong \mathbb{Z}^2$.

Also $b\left(\frac{\alpha_2}{\alpha_1}\right)^2 - bc\left(\frac{\alpha_2}{\alpha_1}\right) + c \leq 0$ only when

$$\frac{bc - \sqrt{bc(bc-4)}}{2b} \leq \frac{\alpha_2}{\alpha_1} \leq \frac{bc + \sqrt{bc(bc-4)}}{2b}$$

which is impossible unless $bc \geq 4$ and for $bc = 4$,

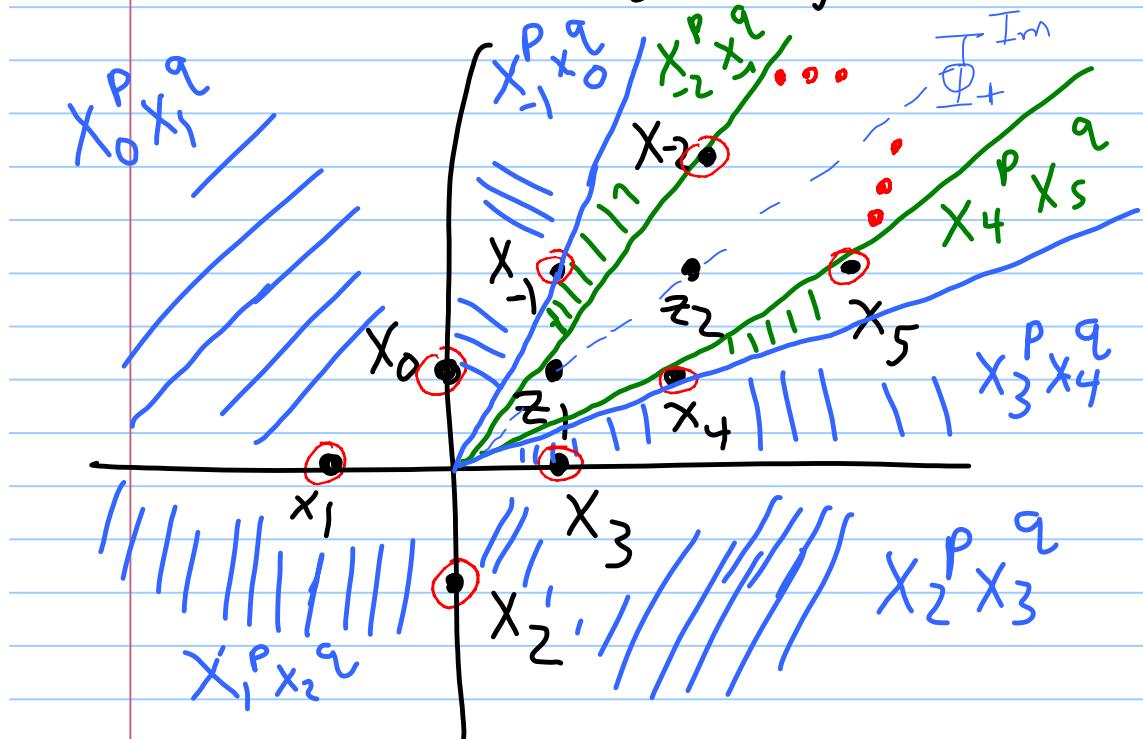
$$(\alpha_1, \alpha_2) = (n, n) \text{ for } (b, c) = (2, 2)$$

$$(\alpha_1, \alpha_2) = (n, 2n) \text{ for } (b, c) = (1, 4).$$

⑨ Recall that in the $(2,2)$ -case, we saw that denominators of cluster variables corresponded to $\alpha_1 + (n+1)\alpha_2$ or $(n+1)\alpha_1 + n\alpha_2$

and $\text{denom}(z_1) \leftrightarrow \alpha_1 + \alpha_2$, T_n has degree $n \Rightarrow \text{den}(z_n) \leftrightarrow n\alpha_1 + n\alpha_2$

Thus Q for $(b,c) = (2,2)$ looks like



Let us prove the Proposition.

Letting $\alpha(m)$ defined by

$$\alpha(1) = -\alpha_1, \quad \alpha(2) = -\alpha_2, \quad \text{For } m \geq 3$$

$$\alpha(m+1) + \alpha(m-1) = \begin{cases} b\alpha(m) & \text{if } m \text{ odd} \\ c\alpha(m) & \text{if } m \text{ even} \end{cases}$$

$$x_m^p x_{m+1}^q \leftrightarrow p\alpha(m) + q\alpha(m+1).$$

(10) Thus suffices to prove

1) For each $m \in \mathbb{Z}$, $\alpha(m)$ and $\alpha(m+1)$ form a \mathbb{Z} -basis of \mathbb{Q} .

2) For each $m \in \mathbb{Z}$, $\alpha(m)$ and $\alpha(m+1)$ are the only positive real roots contained in the additive subgroup $\text{Cone}_m = \sum_{i=0}^{\infty} \alpha(m+i)$.

3) The union $\bigcup_{m \in \mathbb{Z}} \text{Cone}_m = \mathbb{Q} - \bigoplus_{i=1}^{\infty} \mathbb{Z}_+$.

In $bc \leq 3$ (finite type case)
these are checked by inspection.

E.g. A_2 already done above.

B_2 and G_2 cases verified similarly.

Let's assume $bc \geq 4$.

write $\alpha(m) = a_m, \alpha_1 + a_{m2}\alpha_2$, and

note that recurrence \circledast implies

$$\det \begin{pmatrix} a_m & a_{m+1,1} \\ a_{m2} & a_{m+1,2} \end{pmatrix} = \det \begin{pmatrix} a_m & \lambda a_m - a_{m-1,1} \\ a_{m2} & \lambda a_{m2} - a_{m-1,2} \end{pmatrix}$$

with $\lambda = b$ or c

$$= \det \begin{pmatrix} a_{m-1,1} & a_m \\ a_{m-1,2} & a_{m2} \end{pmatrix} \quad ; \quad \begin{matrix} \swarrow [0,1] \\ [-1,0] \end{matrix}$$

$$= \det \begin{pmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{pmatrix} = 1 \quad ; \quad \text{for } m \geq 2.$$

(11) \Rightarrow (1) and also shows that vectors $\alpha^{(m-1)}$ and $\alpha^{(m+1)}$ lie on opposite sides of $\alpha^{(m)}$, implying (2). Also

$\Rightarrow \left\{ \frac{\alpha_{m2}}{\alpha_{m1}} \right\}_{m \geq 3}$ is strictly increasing.

Consider the limit $\lim_{m \rightarrow \infty} \left(\frac{\alpha_{m2}}{\alpha_{m1}} \right)^{u_m}$

Since $\alpha^{(m+2)} = S_1 S_2 \alpha^{(m)}$ for $m \geq 3$,
and $\begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix} = \begin{bmatrix} bc^{-1} & -b \\ c & -1 \end{bmatrix}$, it

follows that $u_{m+2} = \frac{c - u_m}{bc - 1 - bu_m}$,

and the transformation

$u \mapsto f(u) := \frac{c - u}{bc - 1 - bu}$ preserves
the interval $[0, \frac{bc - \sqrt{bc(bc-4)}}{2b}]$ and

satisfies $f(u) > u$ for $0 \leq u \leq \frac{bc - \sqrt{bc(bc-4)}}{2b}$

$$\Rightarrow \lim_{m \rightarrow \infty} \frac{\alpha_{m2}}{\alpha_{m1}} \approx \frac{bc - \sqrt{bc(bc-4)}}{2b})$$

and the $\alpha = a_1 \alpha_1 + a_2 \alpha_2$ with this slope are the positive imaginary roots with smallest slope. An analogous argument works for $m \leq 0$ and the limit vectors are the pos. imag. roots w/ bigest slope.

This implies (2), completing the prop.