

Lecture 11: Rank two positive bases continued

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Note Title

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① Recall that last time we showed a bijection between cluster monomials in a rank two cluster algebra and $\mathbb{Q}-\mathbb{F}_+^{I^m}$. In the $b_c \leq 3$ case, $\mathbb{F}_+^{I^m} = \{\}$.

If $b_c = 4$, $\mathbb{F}_+^{I^m} = \{n\delta : n \in \mathbb{Z}_{\geq 0}\}$
 where $\delta = \begin{cases} \alpha_1 + \alpha_2 & \text{if } (2,2) \\ \alpha_1 + 2\alpha_2 & \text{if } (1,4) \end{cases}$.

To finish the proof of Thm 1

from last time, i.e. In a rank two finite type cluster algebra, the set of cluster monomials form a positive canonical basis, it suffices to show

- 1) The set of cluster monomials are linearly independent.
- 2) Each cluster variable, hence each cluster monomial, is a positive element; that is, they have positive Laurent expansions in terms of any cluster.
- 3) The cluster monomials span $A(b,c)[\text{for } b_c \leq 3]$

③ and 4) If $y \in A(b, c)$ is a linear combination of cluster monomials then each coefficient of this linear combo is equal to some coefficient in the Laurent polynomial expansion of y with respect to some cluster.

We start with (1). we have the following corollary to the proposition at the end of Monday:

Cor: In any cluster algebra of rank 2, the cluster monomials are linearly independent.

PF: Denote $x_1^{g_1} x_2^{g_2}$ as x^γ where $\gamma = g_1\alpha_1 + g_2\alpha_2 \in Q$.

We define a partial order on Q by

$$\gamma_1 \geq \gamma_2 \iff \gamma_1 - \gamma_2 \in Q_+ = \mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2.$$

Since Cluster monomials $\longleftrightarrow Q - \Phi_+^{\text{Irr}}$,

$$\text{for } \alpha \in Q - \Phi_+^{\text{Irr}}, X[\alpha] = X^{-\alpha} + \sum_{\gamma > -\alpha} c_\gamma x^\gamma \quad (*)$$

If

$$\sum_{\alpha \in Q - \Phi_+^{\text{Irr}}} c_\alpha X[\alpha] = 0, \text{ let } S \subset Q - \Phi_+^{\text{Irr}} \text{ be the support of nonzero } c_\alpha's.$$

③ IF $S \neq \emptyset$, pick α s.t. $-\alpha$ is a minimal elt of S .

By $\textcircled{*}$, the Laurent monomial $x^{-\alpha}$ does not occur in any $x[\beta]$ for $\beta \in S - \{\alpha\}$

Thus, no way to cancel out $c_\alpha x^{-\alpha}$. $\Rightarrow \in$

Remark: We will not prove it, but this Corollary can be strengthened to prove that cluster monomials are linearly independent in any cluster algebra of finite type. Proof is analogous, with a different partial order.

Result is conjectured for general cluster algebra.

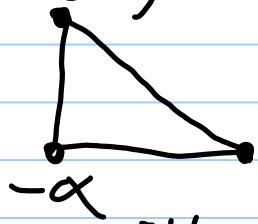
Coming back to rank 2 case, $\textcircled{*}$ can be sharpened.

Def: A Newton polygon of a Laurent polynomial $y \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ is the convex hull in $Q_R = \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2$ of all lattice points γ s.t. x^γ appears with a nonzero coeff in the expansion of y .

Def: Laurent poly y is monic if every Laurent monomial corresponding to a vertex of $\text{Newt}(y)$ appears in the Laurent expansion with a coeff of 1.

④ For $\alpha = q_1\alpha_1 + q_2\alpha_2 \in Q$, let $\Delta(\alpha)$ denote the triangle (possibly degenerate) in Q_R with vertices $-\alpha$, $-\alpha + b\alpha_2$, $-\alpha + c\alpha_1$.

Prop 1: For each positive real root α , the corresponding cluster variable $X[\alpha]$ is a monic Laurent polynomial in x_1 and x_2 with $\text{Newt}(X[\alpha]) = \Delta(\alpha)$.



Note: There is not a higher dimensional version of this prop as it relies on the fact that a convex polygon Π_2 is uniquely determined by Π_1 and $\Pi_1 + \Pi_2$.

Here, $\Pi_1 + \Pi_2$ is the Minkowski sum

$\text{Conv}\left\{ v_i + v_j : v_i \text{ vertex of } \Pi_1 \right\}$.

Prop 1 plus some work, yields the sharpening

Prop 2: Let $X_m^p X_{m+1}^q$ be a cluster monomial containing at least one cl. variable different from x_1 and x_2 . Then $\text{Newt}(X_m^p X_{m+1}^q)$ has empty intersection with the positive quadrant $Q^+ = \mathbb{N}_{\geq 0} \alpha_1 + \mathbb{N}_{\geq 0} \alpha_2$.

E.g. A2 $x_3 = \frac{x_2}{x_1} + \frac{1}{x_1}$, $x_4 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_1 x_2}$, $x_5 = \frac{x_1}{x_2} + \frac{1}{x_2}$

$\text{Newt}(x_3) = \boxed{-}$, $\text{Newt}(x_4) = \boxed{-}$, $\text{Newt}(x_5) = \boxed{-}$

⑤

The positivity of (z) can be proven directly by inspection.

To see (3), we must demonstrate that an arbitrary product of cluster variables can be rewritten as a linear combin. of cluster monomials.

We use the following straightening relations on forbidden products:

We can assume $n \leq \frac{h+2}{2}$ because of the periodicity in the finite type case

A₂: $X_n X_{n+2}$ can be reduced using exchange relation, and

as X, X_4 are also exchangeable ($h=3$)
there is nothing more to show.

$$B_2: X_n X_{n+3} = \begin{cases} X_{n-1} + X_{n+1} & \text{if } n \text{ odd} \\ X_{n+2} + X_{n+4} & \text{if } n \text{ even.} \end{cases}$$

"deg" ≤ 3

$$G_2: X_n X_{n+3} = \begin{cases} X_{n-1} + X_{n+1}^2 & \text{if } n \text{ odd} \\ X_{n+2}^2 + X_{n+4} & \text{if } n \text{ even} \end{cases}$$

"deg" ≤ 6

$$X_n X_{n+4} = \begin{cases} X_{n-2} + X_{n+2} + 3 & \text{if } n \text{ odd} \\ X_{n-2} + X_{n+2} & \text{if } n \text{ even} \end{cases}$$

"deg" = 6 or 10 "deg" ≤ 5

⑥ In the B_2 case, take

$$\text{degree}(x_n) = \begin{cases} 3 & \text{if } n \text{ odd} \\ 2 & \text{if } n \text{ even} \end{cases}$$

In G_2 case,

$$\text{degree}(x_n) = \begin{cases} 5 & \text{if } n \text{ odd} \\ 3 & \text{if } n \text{ even} \end{cases}$$

\Rightarrow degree strictly decreases as these straightening rules applied so linear comb. of cluster monomials has smaller "degree" than the original product.

We finish the proof (and prove (4)):

Suppose $y \in A(b, c)$ is a linear combination of cluster monomials, $y = \sum c_{n,p,q} x_m^p x_{m+1}^q$.

Since seeds are equivalent to one another in the rank two case, we consider the Laurent expansion of y with respect to cluster $\{x_1, x_2\}$ w.l.o.g.

⑦ Claim: For $p, q \geq 0$, the coeff of $x_1^p x_2^q$ in linear combination $\textcircled{*}$ equals coeff of $x_1^p x_2^q$ in the Laurent polynomial expansion of y with respect to $\{x_1, x_2\}$.

Pf of Claim: Can write $y = \lambda x_1^p x_2^q + \dots$ other terms
and we rewrite all cluster monomials comprising the (other terms) in terms of cluster $\{x_1, x_2\}$.

However, by Prop 2, if $m \neq 1$, $\text{Newt}(x_m^r x_{m+1}^s) \cap Q_+ = \emptyset$.

\Rightarrow none of the other terms will have $x_1^p x_2^q$ in its Laurent expansion.

\Rightarrow coefficient λ is the same in both expansions.

$\Rightarrow \beta = \{\text{cluster monomials}\}$ is a basis, each element of which is a positive element, and from (4), it follows that if y is not a positive linear combo of cluster monomials, then y could not be a positive element.

Completes proof of Thm 1.

(8)

Proofs for $b_c=4$ (affine types)

We want to prove Thm 2, which states that $\mathcal{B} = \{\text{cluster monomials}\} \cup \{T_n(z) : n \geq 1\}$ is a positive canonical basis.

Need to show:

- 1) The set \mathcal{B} is linearly independent.
- 2) Each cluster variable x_n , and each $z_n = T_n(z)$ is a positive element.
- 3) The set \mathcal{B} spans $A(b, c)$ for $b_c = 4$.
- 4) If $y \in A(b, c)$ is a linear combination of cluster monomials and z_n 's then each coefficient of this linear combo is equal to some coefficient in the Laurent polynomial expansion of y with respect to some cluster.

For (1) we already have that cluster monomials' are linearly independent.

Letting $x[n\delta] = z_n$, where $\Phi_+^{I_m} = \{n\delta : n \geq 1\}$, adaptation of above proof gives that all $\mathcal{B} = \{\text{cluster monomials}\} \cup \{z_n\}$ lin ind.

⑨ We will come back to positivity (2).

For (3), we must reduce the forbidden products $z_p z_r, z_p x_n, x_n x_{n+l}$ where $p, r > 0, n \in \mathbb{Z}, l \geq 2$.

Prop: If $b_c = 4$ and $p \geq r \geq 1$, we have

$$z_p z_r = z_{p-r} + z_{p+r}$$

[we use the convention $z_0 = \bar{z}_0(z) = 2$]

In the $(b_j c) = (2, 2)$ case, $p \geq 1, n \in \mathbb{Z}$,

$$z_p x_n = x_{n-p} + x_{n+p}$$

$$x_n x_{n+l} = x_{\lfloor n + \frac{l}{2} \rfloor} x_{\lceil n + \frac{l}{2} \rceil} + \sum_{k=1}^{\lfloor l/2 \rfloor} k z_{n-2k}$$

We omit the straightening relations for the $(b_j c) = (1, 4)$ case.

Note that $x_{\lfloor n + \frac{l}{2} \rfloor} x_{\lceil n + \frac{l}{2} \rceil}$ is a cl. mon.,

and so all elements on the RHS are in \mathcal{B} .

These relations can be proven algebraically.
Later in the course, we will see a geometric approach.

(10)

This will also take some explanation
so we come back to it.

To prove (4) in the affine case,
we begin by noting that if $y = \lambda x_1^p x_2^q + \dots$
then λ must also be the coefficient
of y in cluster $\{x_1, x_2\}$ for the same
reason as before. By symmetry, this
holds for all coefficients of cluster monomials
in the expansion of y .

We wish to also show that

Claim: If $y = \lambda z_p + \dots$, then

$y = \lambda x_1^p x_2^{-p} + \dots$ in the expansion
in cluster $\{x_1, x_2\}$.

PF of claim: $z = \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{1}{x_1 x_2}$ in $\{x_1, x_2\}$

T_p has degree p , so $z_p = \frac{x_1^p}{x_2^p} + \dots$

Using Newton polygons, possible to see
that

$\frac{x_1^p}{x_2^p}$ does not occur in the expansion

of z_n unless $n=p$, and a degree argument
 $\Rightarrow \frac{x_1^p}{x_2^p}$ does not appear in the expansion

of any x_n .

Claim then proven as before

II Positivity for affine rank two

One can prove this from straightening relations and facts about Chebyshev polynomials.

Straightening allows us to write

$X_n(X_1^{n-2} X_2^{n-3})$ as a positive expansion

of other x_m 's and z_p 's, and by induction, the x_m 's are positive Laurent polynomials.

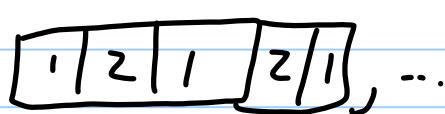
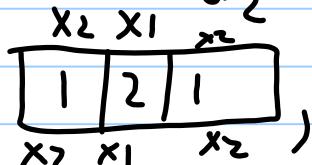
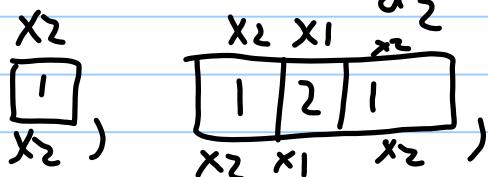
In fact, knowing the denominators we will be able to rewrite for bidden product

$X_n(X_1^{n-2} X_2^{n-3})$ as a positive polynomial in x_1, x_2 .

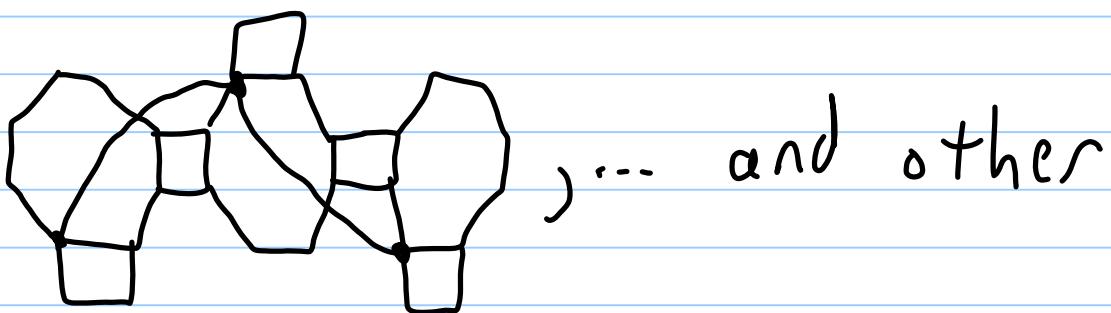
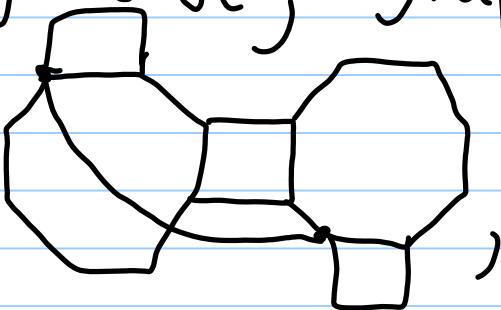
Jim Propp and I together also gave a combinatorial proof of positivity by finding two infinite families of graphs such that

$$X_n = \sum_{\text{Perf Matching } P \text{ of } G_n} \frac{x(P)}{x_1^{d_1} x_2^{d_2}}$$

where $d_1 = n-2$ in $(2,2)$ case



(12) For $(1,4)$ case, graphs look like



combinatorial types made up of octagons and squares.

We will see later in the course how to think of cluster algebras of rank two affines as examples of cluster algebras from surfaces.

Recent work by Giovanni Cerulli Irelli (arxiv July 2010) describes the positive canonical basis for

type $A_2^{(1)}$. One such representative quiver for this type is

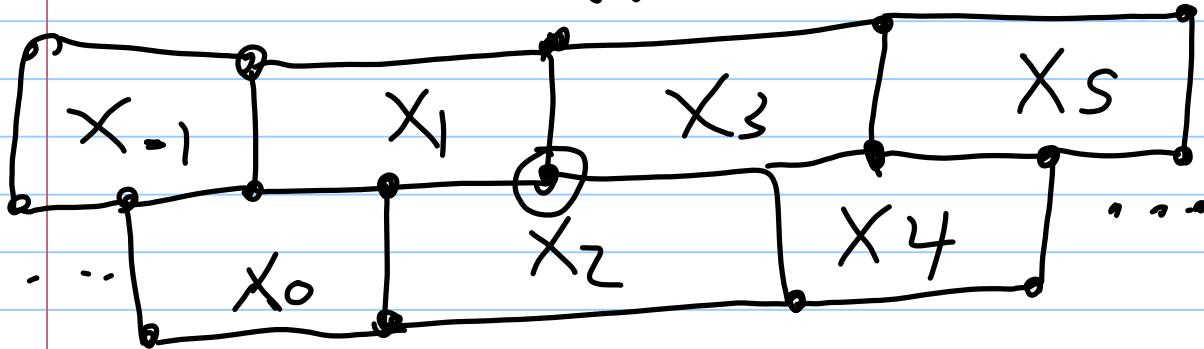


(13) This example also appears in
FZ Cluster Algebras 1 with cluster
variables

$$\{x_n : n \in \mathbb{Z}\} \cup \{w, z\}, x_n x_{n-3} = x_{n-1} x_{n-2} + 1,$$

and exchange graph

w



z

$$w = \frac{x_1 + x_3}{x_2}, z = \frac{x_1 x_2 + x_2 x_3 + 1}{x_1 x_3}$$

Thm [Incoll] For $A_2^{(1)}$

$$B = \{\text{cluster mons}\} \cup \{u_n w^k, u_n z^k\}_{n \geq 1, k \geq 0}$$

(where $u_0 = 1, u_1 = zw - 2, u_2 = u_1^2 - z, u_{n+1} = u_1 u_n - u_{n-1}$)

is a positive canonical basis.