

Lecture 17: Introduction to Quiver RePs (2-28-11) Math 8680 Gregg Musiker

Note Title

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① One last comment about positive
canonical bases: (1102.3050)

Very recent! (Feb 18, 2011) on arxiv,
again by Giovanni Trelle, that
proves $B = \{\text{cluster monomials}\}$ is a
positive canonical basis for cluster
algebras of type A, D, E .

Method uses quivers with potentials,
outside scope of this course but we
may approach this topic when discussing
connection between quivers and cluster
algebras later on.

But for now, we switch gears:

Note: Unless otherwise specified, we let K be any field. (Sometimes will require K alg closed or $\text{char } K = 0$, but for now we do not include any restrictions.)

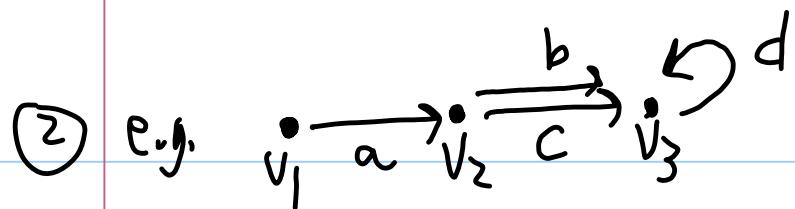
Def: A quiver Q is a finite directed graph. More specifically, $Q = (Q_0, Q_1)$

where $Q_0 =$ a finite set of vertices,

$Q_1 =$ a finite set of arrows

With functions $s: Q_1 \rightarrow Q_0$ (source)

$t: Q_1 \rightarrow Q_0$ (target)



$$Q_0 = \{v_1, v_2, v_3\} \quad Q_1 = \{a, b, c, d\}$$

$$s(a) = v_1, \quad t(a) = v_2, \quad \text{etc.}$$

Notice that loops and multiple edges are perfectly allowed, as are 2-cycles

Disconnected quivers are also ok.

Def: Let Q be a quiver. A representation V of Q is an assignment of a finite dimensional vector space $V(x)$ over base field K for each $x \in Q_0$, and for each arrow $a \in Q_1$, a linear transformation

$$V(a): V(sa) \rightarrow V(ta).$$

As is common in representation theory, we study subrepresentations and morphisms between repns.

Example: Let $Q = \begin{array}{c} 1 \\ \xrightarrow{a} \\ 2 \end{array}$. We can describe a repn of Q as a choice of 2 finite dim-vector spaces $/K$,

$$V(1) = T, \quad V(2) = U, \quad \text{and a linear transf.}$$

$$V(a) = \phi: T \rightarrow U.$$

If we pick bases for vec spaces T and U s.t. $T \cong K^m$, $U \cong K^n$ then lin transf. ϕ corresponds to an $n \times m$ matrix.

③ Def: Fix Q to be a quiver and let
 V and W to be reps of Q .

A morphism $\phi: V \rightarrow W$ consists of linear transformations $\{\phi(x): V(x) \rightarrow W(x) \text{ for } x \in Q_0\}$
 s.t. for every arrow $a \in Q_1$, we have
 a commutative diagram

$$\begin{array}{ccc} V(sa) & \xrightarrow{V(a)} & V(ta) \\ \phi(sa) \downarrow & & \downarrow \phi(ta) \\ W(sa) & \xrightarrow{W(a)} & W(ta) \end{array}$$

In other words, $w(a)\phi(sa) = \phi(ta)v(a)$
 for all $a \in Q_1$.

Moreover, we say that $\phi: V \rightarrow W$ is an isomorphism if all of the $\phi(x)$'s are isomorphisms as linear transformations, i.e. bijective.

Back to Example 1 $Q = \begin{matrix} & \xrightarrow{a} \\ \bullet & \xrightarrow{b} \bullet \end{matrix}$

$V(1) \cong k^n$, $V(2) \cong k^m$, $V(a) = M$,
 an $n \times m$ matrix in these bases.

Let $W(1) \cong k^{n'}$, $W(2) \cong k^{m'}$, $W(a) = N$.

$\phi: V \rightarrow W$ is a morphism only if

$$N \phi(k^n) = \phi(k^m) M$$

and is an isomorphism $\Leftrightarrow n' = n, m' = m$
 and $\phi(k^n)$ is an invertible $n \times n$ matrix P ,
 $\phi(k^m)$ is an "m \times m" "Q".

(4) $\Rightarrow N = QMP^{-1}$, i.e. M and N have the same rank.

Example 2a

$$\text{Let } Q = \begin{pmatrix} & a \\ 1 & & b \\ & 2 & \\ & 3 & \end{pmatrix}$$

$$V = K \xleftarrow{1} K \xrightarrow{1} K$$

$$W = K \xleftarrow{1} K \xrightarrow{0} 0$$

A morphism $\phi: V \rightarrow W$ is determined by a choice of scalar $\alpha \in k$ and

$$\begin{array}{c} V = K \xleftarrow{1} K \xrightarrow{1} K \\ \phi \downarrow \quad \downarrow \alpha \quad \downarrow \alpha \quad \downarrow \alpha \\ W = K \xleftarrow{1} K \xrightarrow{1} K \end{array}$$

Claim: For first square to commute need scalar α to be same on both verticals.

$$\text{e.g. } K \xleftarrow{1} K$$

$$\begin{matrix} \alpha \downarrow & & \downarrow \beta \\ & & \\ K & \xleftarrow{1} & K \end{matrix} \quad \begin{matrix} \text{does not commute} \\ \text{if } \alpha \neq \beta \end{matrix}$$

Similarly the rightmost vertical zero map is also forced.

Example 2b Now let us consider morphisms

$$\phi: W \rightarrow V$$

$$W \quad K \xleftarrow{1} K \xrightarrow{0} 0$$

$$\begin{cases} \phi \\ \downarrow \alpha \\ V \quad K \xleftarrow{1} K \xrightarrow{1} K \end{cases}$$

The rightmost vertical map must be a zero, which forces a zero in the middle and then the left.

⑤ Example 3 $Q = \begin{smallmatrix} & a \\ ; & \end{smallmatrix}$

Given repn's V, W of Q , when are they isomorphic?

V determined by $V(1) \cong k^n$, $V(a):V(1) \hookrightarrow$, an $m \times m$ matrix M

$W(1) \cong k^n$, $W(a) \cong N$, $n \times n$ matrix.

Claim: If $V \cong W$, then $m=n$ and there is an isomorphism $\phi: k^n \rightarrow k^n$ corresponding to $n \times n$ invertible matrix P so that $PMP^{-1} = N$.

In other words $V(a)$ and $W(a)$ are similar matrices when V and W are isomorphic repns of $\begin{smallmatrix} & a \\ ; & \end{smallmatrix}$

Def: Let V be a representation of Q . we say that the dimension vector of V is $\underline{\dim} V = (\dim_K V(1), \dim_K V(2), \dots, \dim_K V(n))$.

Instead of talking about classifying all representations up to isomorphism, it is more useful to only consider those reps with the same dimension vectors.

Remark: If $\underline{\dim} V \neq \underline{\dim} W$, then $V \not\cong W$.

Example 4: Let $Q = \begin{smallmatrix} & a_1 & & a_{n-1} \\ ; & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdots & \cdot & \rightarrow & \cdot \\ & 2 & & 3 & & & & & & & n \end{smallmatrix}$

Let V, W be reps with $V(x) = k, W(x) = k$ for all $x \in Q_0$. (i.e. $\underline{\dim} V = \underline{\dim} W = (1, 1, \dots, 1)$).

Exercise: Characterize when $V \cong W$.

⑥ The answer will be given next class.

Def: Let Q be a quiver. W a rep.
A subrepresentation of W is a rep V such that $V(x) \subseteq W(x)$ for all $x \in Q_0$ and for all $a \in Q_1$, $\boxed{V(a)|_{V(sa)} \subseteq V(ta)}$

Here the notation $L|_V$ denotes the range of lin. transf. L restricted to domain V .

Def: The zero rep, let us denote it as Z , is defined as $Z(x)=0 \quad \forall x \in Q_0$ and $Z(a)=0 \quad \forall a \in Q_1$.

Def: Let Q be a quiver and V a rep of Q . Then V is simple (equiv. irreducible) if the only subreps of V are V and Z .

Example Let Q be a quiver, $x \in Q_0$.

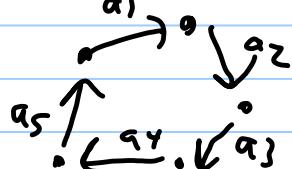
Define E_x by $E_x(x)=k$,

$$E_x(y)=0 \quad \forall y \in Q_0, \\ E_x(a)=0 \quad \forall a \in Q_1, \quad y \neq x,$$

Then E_x is a simple rep.

Def: Let Q be a quiver. An oriented cycle in Q is a sequence a_1, a_2, \dots, a_n of arrows so that

$$t(a_i) = s(a_{i+1}) \text{ for } 1 \leq i \leq n-1, \text{ and} \\ t(a_n) = s(a_1).$$



⑦ Just to clarify: $Q = \begin{array}{c} \circ \xrightarrow{\cdot} \circ \\ \nwarrow \quad \downarrow \\ \circ \end{array}$ has an oriented cycle, but $Q' = \begin{array}{c} \circ \xrightarrow{\cdot} \circ \\ \downarrow \quad \swarrow \\ \circ \end{array}$ does not.

Prop: If Q has no oriented cycles and vertices $\{1, 2, \dots, n\}$, then E_1, E_2, \dots, E_n are the only simple reps for Q up to isomorphism.

PF: Assume otherwise and let V be a rep of Q and Q' be the subquiver on which V has nonzero vector spaces. Since $V \neq E_1, \dots, E_n$, or 0 , Q' has at least two vertices. Q' not nec. connected but also contains no oriented cycles. Thus Q' has at least one source, x , and let V' be defined as $V'(x) = V(x)$, $V'(y) = 0$ for $y \neq x$. V' is not V nor the 0 -rep, and V' is a valid subrep as vector space $V(x)$ was not the target of any nonzero maps. $\Rightarrow V$ not simple.

Point: For Q with no oriented cycles, the classification of simple/irreducible representations is essentially trivial.

This differs greatly from repn theory of finite groups. E.g., for symmetric groups, irreducible representations classified by partitions and constructed using Specht modules or Young symmetrizers.

Nontrivial example: Let $Q = \begin{array}{c} \circ \xrightarrow{a} \circ \\ ; \quad \swarrow \quad \searrow \\ \circ \end{array}$
 (oriented 2-cycle) Let $V_\lambda = K \xrightarrow{\lambda} K \xleftarrow{\lambda} K$ for $\lambda \in K, \lambda \neq 0$.

(8)

Claim: $\{V_\lambda : \lambda \neq 0\} \cup \{E_1, E_2\}$ are the only simple reps of this Q up to isomorphism.

Demonstration: $O \xrightleftharpoons[\text{because would need } \text{Im}(\lambda) = O]{\lambda} K$ is not a subrep because we would need $\text{Im}(\lambda) = O$, but $\lambda \neq 0$.

Similarly $K \xrightleftharpoons[\text{O}]{\lambda} O$ is not a subrep for the same reason. In particular if we let either $W(1)$ or $W(2) = O$ then $W(1) = W(2) = O$ and we get the zero rep as the only smaller subrep.

Also get $V' = K \xrightleftharpoons[\text{same reason}]{\mu, \lambda \neq 0} K$, $\mu, \lambda \neq 0$ is a simple rep'n for the same reason.

However $V' \cong V_\lambda$. If μ or λ were 0, we would get V' is not simple.

$(K \xrightleftharpoons[\text{O}]{\lambda} K \text{ has subrep } O \xrightleftharpoons[\text{O}]{\lambda} K)$

Finally, if we consider higher dimensional reps, such as $V = K^n \xrightleftharpoons[\text{P}]{\phi} k^m$,

with $n \geq 2, m \geq 1$, then we can project to lower dimensional vector spaces and get

$$V' \subset V, \quad V' \cong K^{n-1} \xrightleftharpoons[\varphi|_{k^{m-1}}]{\phi|_{k^{n-1}}} k^{m-1}.$$

$\Rightarrow V$ not simple.

Even though we could say something nontrivial for simple reps for some quivers with oriented cycles, it will turn out indecomposable reps are more interesting to study.

⑨ Def: Let Q be a quiver, and let V, W
be reps of Q . Then $V \oplus W$ is
a rep of Q defined by

$$(V \oplus W)(x) = V(x) \oplus W(x) \quad \forall x \in Q_0 \\ = \{(v_j w_j) \mid v \in V(x), w \in W(x)\},$$

$$(V \oplus W)(a) = V(a) \oplus W(a) \quad \text{for } a \in Q_1 \\ \text{The map } (V, W) \mapsto (V(\cdot)|_V, W(\cdot)|_W)$$

Def: A rep V of Q is called
decomposable if $V \cong U \oplus W$
for some nonzero reps U, W of Q ,

[i.e. for some [not nec. all] vertices,
the $V(x)$ break into nontrivial direct sums]

V is indecomposable if it is not
decomposable.

Lemma: Let Q be a quiver. IF
 V is a simple rep of Q , then
 V is indecomposable.

Pf: If V is simple but decomposable,
i.e. $V \cong U \oplus W$ for U, W nonzero
reps then $U \oplus W$ would be simple.

However, $U \oplus W$ is a subrep of V
and $U \oplus W \neq \text{zero rep}$, and $\neq V$. \square

Remark: Converse is false!!
for quiver representations. (Even
though true for finite groups.)

(10) Example $Q = \begin{array}{c} \bullet \\[-1ex] \xrightarrow{a} \\[-1ex] \bullet \end{array}$

$V = k \xrightarrow{a} k$ is not simple because it has $k \xrightarrow{0} 0$ as a subrep. [In fact Q has no oriented cycles and $V \not\cong E_1, E_2$ so we will be able to conclude from earlier prop that V is not simple.]

However, V is indecomposable

Might try $V \cong U \oplus W$ with

$U = k \xrightarrow{a} 0, W = 0 \xrightarrow{a} k$
but then both $U(a) = W(a) = 0$

and $(U \oplus W)(a) = 0 \oplus 0 = 0$

so not isomorphic to V , with $V(a) = 1$.

Another Example: Same Q

$$V' = k^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} k^2$$

$$V' \cong k^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} k^2 \text{ so } V' \cong V \oplus V.$$

Lemma: Let V be a rep of Q . Then $V \cong V_1 \oplus V_2 \oplus \dots \oplus V_m$ for some indecomposables V_i .

Pf: If V is not indecomposable,

$V \cong U \oplus W$ where U and W are both

(11) not zero reps. If one of the U or W is not indecomp, we can continue this way. Process must terminate since dimensions strictly decreasing.

Maschke's Thm for finite groups

Any rep V can be written as a unique direct sum of simples up to perm. of factors and isomorphism.

Called semisimple or complete reducibility,

For quiver reps have decomposition into indecomposables and though not mentioned above, this direct sum is also unique up to permutation and isomorphism.

Krull-Remak-Schmidt Theorem

We will prove next week.

Def: Q is of finite representation type if Q admits a finite # of indecomposables, up to isomorphism.

E.g. $\begin{array}{ccc} & a & \\ \overset{\circ}{\underset{\circ}{\longrightarrow}} & \overset{\circ}{\underset{\circ}{\longrightarrow}} & \end{array}$ is finite rep type w/ only indecomps (up to isom) are E_1, E_2 , and $\begin{array}{ccc} & 1 & \\ \overset{\circ}{\underset{\circ}{\longrightarrow}} & \overset{\circ}{\underset{\circ}{\longrightarrow}} & \end{array}$.

Exercise (Answer next time)

Show that $Q = \begin{array}{ccc} & a & \\ \overset{\circ}{\underset{\circ}{\longrightarrow}} & \overset{\circ}{\underset{b}{\longrightarrow}} & \end{array}$ is not of finite representation type.