

# Lecture 13: The Path Alg. of a Quiver

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Note Title

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① Let us start today with the exercises from last time and further examples.

$$Q = \begin{array}{ccccccc} & & a_1 & & a_2 & & \dots & & a_{n-1} & & \\ & & \rightarrow & & \rightarrow & & \dots & & \rightarrow & & \\ \bullet & & \bullet & & \bullet & & \dots & & \bullet & & \bullet \\ 1 & & 2 & & 3 & & \dots & & n-1 & & n \end{array}$$

Let  $\underline{\dim} V = \underline{\dim} W = (1, 1, \dots, 1)$

Characterize when  $V \cong W$

Since  $V(i)$  and  $W(i)$  both  $\cong K$   
for  $i \in \{1, 2, \dots, n\}$

up to isomorphism,  $V(a_i) \cong 0$  or  $I$   
and similarly  $W(a_i) \cong 0$  or  $I$  map.

Also if  $\varphi: V \rightarrow W$  is an isomorphism,  
 $\varphi(i): V(i) \rightarrow W(i)$  is scalar

mult by  $\alpha_i \neq 0$  for each  $i$ .

Since we need the squares to commute

$$\begin{array}{ccc} & \xrightarrow{1 \text{ or } 0} & \\ \alpha_i \downarrow & & \downarrow \alpha_{i+1} \\ & \xrightarrow{1 \text{ or } 0} & \end{array}$$

commutes only if  
both horizontal  
maps are 1 or  
both are 0 and  
all  $\alpha_i$ 's are equal.

② In general, if  $\dim V = \dim W$ ,

$$\begin{array}{ccc}
 V(i) \cong K^m & \xrightarrow{V(a_i)} & V(i+1) \cong K^n \\
 \varphi(i) \downarrow & & \varphi(i+1) \downarrow \\
 W(i) \cong K^m & \xrightarrow{W(a_i)} & W(i+1) \cong K^n
 \end{array}$$

This square commutes for some choice of isomorphisms  $\varphi(i)$  and  $\varphi(i+1)$  if and only if

$V(a_i)$  and  $W(a_i)$  have the same ranks as matrices.

Thus ranks of the  $V(a_i)$ 's and  $\dim V$  determine  $V$  up to isomorphism.

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Exercise: Show that  $Q = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}_2$  is not of finite rep type.

We can actually construct for each

$$V, \dim V = (n, n), \quad K^n \xrightarrow{I_n} K^n \\
 \downarrow J_n(\lambda)$$

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ 0 & & & \lambda \end{bmatrix}$$

③ Example:  $Q = \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ a \quad b \end{array}$  For each  $n \geq 1$ ,  
 let  $V$  be the rep defined by  $V_n(i) = K^n$ ,

$$V_n(a) = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}, \quad V_n(b) = \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

Claim:  $V_n$  is simple.

If we restrict to a subspace  $U$ , then

$\text{Im}(V_n(a)|_U)$  and  $\text{Im}(V_n(b)|_U)$  cannot both be in  $U$ .

Thus, there are higher dimensional simples for this quiver.

On the other hand, if we consider  $Q = \begin{array}{c} \bullet \\ \rightarrow a \\ i \end{array}$

and  $K$  is algebraically closed, then any simple  $Q$ -rep must be one-dimensional.

If  $V$  is a higher dim  $Q$ -rep /  $K$ , then we can find an eigenvector  $v \in V(i)$  for linear transformation  $V(a)$ .

If  $K = \mathbb{R}$ , and  $V(i) = K^2$ ,  $V(a) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  is simple.

More about  $Q = \bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \rightarrow \dots \xrightarrow{a_{n-1}} \bullet$

④ Let us classify the set of indecomposables up to isomorphism.

Let  $I_{i,j}$  be the rep (for  $1 \leq i \leq j \leq n$ )

$$0 \rightarrow \dots \rightarrow 0 \rightarrow \underset{\text{ith}}{k} \xrightarrow{1} k \xrightarrow{1} \dots \xrightarrow{1} k \rightarrow 0 \rightarrow \dots \rightarrow 0$$

If  $i=j$ ,  $I_{i,i} = E_i$  (simple rep)

Claim:  $I_{i,j}$  is indecomposable for all  $1 \leq i \leq j \leq n$ .

Step 1: Let  $V$  be a subrep of  $I_{i,j}$ .

If  $V(k) = 0$ ,  $V(i) \neq 0$ , and  $i < k \leq j$ , and we have

$k \xrightarrow{1} 0$  where we used to  $k \xrightarrow{1} k$ , then this cannot be a valid subrep as  $\text{im}(1|_k) \neq 0$ .

However  $I_{i+l,j}$  for  $i \leq i+l \leq j$  is a subrep of  $I_{i,j}$ .

[I.e., can turn  $k$ 's into  $0$ 's at beginning of the string, but not the end.]

So  $I_{i,j}$ 's are not simple if  $i \neq j$ .

However they are indecomposable because if  $I_{i,j} \cong T \oplus U$  then note that  $T, U$  would have to be subreps of  $I_{i,j}$ .

⑤ But  $T(j) = k \not\cong U(j) = k$  in that case  $\Rightarrow (T \oplus U)(j) = k^2$  not  $k$ .  $\square$

Claim:  $I_{i,j}$ 's are only indecomposables of  $Q$ .

Consider  $V$  where  $\max(\dim_k V(x) : x \in Q_0) > 1$  and let  $x$  be the leftmost vertex s.t.  $\dim_k V(x) > 1$ . Then we can restrict to a subspace  $U \subseteq V(x)$  containing image of  $V(a)$  where  $\begin{matrix} \bullet & \xrightarrow{a} & x \\ x^{-1} & & \end{matrix}$  (Note:  $\dim_k[\text{Im } V(a)] \leq 1$ )

We then restrict to the appropriate subspaces to the right. This gives us a valid subrep  $U \subseteq V$  and  $V = U \oplus U^\perp$ .

If we now consider a rep  $V$  where all  $\dim_k(V(x)) = 0$  or  $1$ , then  $V$  either contains a  $k \xrightarrow{a} k$  or  $k \xrightarrow{a} 0 \xrightarrow{b} k$ .

Either way, we can decompose into two nontrivial reps where the two vec spaces  $\cong k$  are separated.  $\square$

Thus,  $I_{i,j}$ 's are only indecomposables.

(Does anyone recognize this classification?)

Cor:  $Q = \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \dots \rightarrow \bullet$  is of finite representation type.

## ⑥ New Topic (The path algebra)

Def: A path  $p$  is a sequence  $a_1, a_2, \dots, a_m$  of arrows such that  $sa_i = ta_{i+1}$  for  $i=1, \dots, m-1$ . ( $s = \text{source}$ ,  $t = \text{target}$ )

Think of path running from right to left.

$$p = \bullet \xleftarrow{a_1} \bullet \xleftarrow{a_2} \dots \bullet \xleftarrow{a_{m-1}} \bullet \xleftarrow{a_m}$$

For a path  $p$ , we define source  $sp = sa_1$  and target  $tp = ta_m$ .

For every  $x \in Q_0$ , we also let  $e_x$  denote the trivial "lazy" path  $se_x = te_x = x$ .

The concatenation of two paths

$$p = a_1 \dots a_m \quad \& \quad q = b_1 \dots b_n \quad \text{is}$$

$$pq = a_1 \dots a_m b_1 \dots b_n.$$

$$(e_{tq})q := q, \quad p(e_{sp}) := p.$$

$$\begin{array}{ccc} \xleftarrow{p} \bullet & & \bullet \xleftarrow{q} \\ e_{sp} & & e_{tq} \end{array}$$

and  $e_x e_x = e_x$  for all  $x \in Q_0$   
 $e_x e_y = 0$  if  $x \neq y$ .

⑦ Let  $K$  be a field. An (associative)  $K$ -algebra  $A$  is a ring (rings have 1) which is also a  $K$ -vector space s.t.

$$(\lambda \cdot a)b = (a)(\lambda \cdot b) = \lambda \cdot (ab) \text{ for all } a, b \in A \text{ and } \lambda \in K$$

[ $\cdot$  = scalar mult as vec space]

Example: If  $\{v_i\}$  is a basis for  $K$ -vec space  $V$  then defining an associative products on basis elements as  $v_i * v_j = w_{ij}$  for  $w_{ij} \in V$ .

Then we can extend linearly as

$$t = \sum_k a_k v_k, \quad u = \sum_l a_l v_l,$$

$$t * u = \sum_k \sum_l a_k a_l (v_k * v_l)$$

and turn  $V$  into a  $K$ -algebra.

Example:  $V$  is the vector space over  $K$  with basis  $\{t^0, t^1, t^2, \dots\}$ ,  $t^m * t^n = t^{m+n}$ .

This gives an algebraic structure on all of  $V$ , which is isomorphic to  $K[t]$ .

The path algebra of a quiver is another example of  $K$ -algebra  $KQ$

$$p * q := \begin{cases} \text{concatenation } pq & \text{if } sp = tq \\ 0 & \text{otherwise} \end{cases}$$

⑧ This multiplication is clearly associative.

Examples:  $Q = \begin{matrix} & & a \\ & \circ & \swarrow \\ & i & \\ & & \end{matrix}$

$KQ$  has paths  $\{e_1, a, a^2, a^3, \dots\}$   
 $\parallel$   
 $aa$

$e_1 * a = a * e_1 = a$  and  $a^i * a^j = a^{i+j}$ .

$\Rightarrow K(\begin{matrix} & & a \\ & \circ & \swarrow \\ & i & \\ & & \end{matrix}) \cong K[t]$   $e_1 \mapsto 1$   
 $a \mapsto t$

$Q = \begin{matrix} & & a \\ & \circ & \swarrow \\ z & & i \\ & & \end{matrix}$  paths are  $\{e_1, e_2, a\}$

Mult table

	$e_1$	$e_2$	$a$
$e_1$	$e_1$	$0$	$0$
$e_2$	$0$	$e_2$	$a$
$a$	$a$	$0$	$0$

$KQ \cong B_2$ , the space of lower-triang.  $2 \times 2$  matrices.

$B_2$  has basis  $\begin{matrix} \uparrow \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ e_1 \end{matrix}, \begin{matrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ e_2 \end{matrix}, \begin{matrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ a \end{matrix}$

Multiplication Table of  $B_2$  is the same as  $K(\begin{matrix} & & a \\ & \circ & \swarrow \\ z & & i \\ & & \end{matrix})$ .

Exercise :  $K(\begin{matrix} & & \circ & \leftarrow & \circ & \leftarrow & \dots & \leftarrow & \circ \\ n & & n-1 & & & & & & 1 \end{matrix}) \cong B_n$ , lower triangular  $n \times n$  matrices.

⑨ Let  $Q = \begin{array}{c} a \quad b \\ \curvearrowright \quad \curvearrowleft \\ \bullet \\ \downarrow \\ 1 \end{array}$ ,  $KQ \cong K\langle a, b \rangle$

Free associative  $K$ -algebra on two gens  $\{a, b\}$ .

Some properties of  $KQ$

• For each  $x \in Q_0$ ,  $e_x$  is idempotent  
 $e_x e_x = e_x$ .

• For  $x, y \in Q_0$ ,  $x \neq y$   $e_x e_y = 0 = e_y e_x$   
so  $e_x$  and  $e_y$  are orthogonal.

• For each  $x \in Q_0$ , we cannot express  
 $e_x = r + s$  where  $r, s \in KQ$  are a pair of  
orthogonal idempotents. (Proof Next time)

• If  $Q_0 = \{1, 2, \dots, n\}$  then  $e_1 + e_2 + \dots + e_n = \text{id}$   
in  $KQ$

$$e_i \cdot p = \begin{cases} p & \text{if } tp = i \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Similarly, } p \cdot e_i = \begin{cases} p & \text{if } sp = i \\ 0 & \text{o.w.} \end{cases}$$

$$\Rightarrow p(e_1 + \dots + e_n) = p = (e_1 + \dots + e_n)p$$

Prop: Let  $KQ$  be the path algebra of a  
given  $Q$ .  $KQ$  is finite dim /  $K$   
if and only if  $K$  has no oriented cycles.

(oriented cycles are paths w/  $sp = tp$ )