

Lecture 15: Projectives and Injectives

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(3-9-11)

Note Title

3/7/2011

① Follows Section 3 of M. Barot
"Representations of Quivers"

Let \mathcal{C} be a category. A morphism $f: X \rightarrow Y$ is called a monomorphism if for every two morphisms $g_1, g_2: W \rightarrow X$ we have

$$f \circ g_1 = f \circ g_2: W \rightarrow Y \Rightarrow g_1 = g_2.$$

[Universal property/category definition of an injective homomorphism]

Let \mathcal{C} be a category. A morphism $f: X \rightarrow Y$ is called a epimorphism if for every two morphisms $h_1, h_2: Y \rightarrow Z$ we have

$$h_1 \circ f = h_2 \circ f: X \rightarrow Z \Rightarrow h_1 = h_2.$$

[Universal property/category definition of a surjective homomorphism]

Def: If $f: M \rightarrow N$ is a morphism of left R -modules, then

$$\ker f = \{x \in M \mid f(x) = 0\} \subseteq M$$

$$\text{coker } f = N/f(M) \subseteq N. \quad \text{Im } f = f(M)$$

The $\ker f: V \rightarrow W$ between two quiver representations is the subrep K of V s.t. $K(x) = \ker f|_{V(x)} \quad \forall x \in Q_0$ and $K(a): \ker f|_{V(sa)} \rightarrow \ker f|_{V(ta)}$.

(2)

The $\text{coker } f$ & $\text{Im } f$ are similarly defined subreps of W_0 .

Note: $f: V \rightarrow W$ injective $\Leftrightarrow \ker f$

is the zero rep

and $f: V \rightarrow W$ surjective $\Leftrightarrow \text{Im } f$
is the full rep W .

\Leftarrow $\text{coker } f$ is the zero rep.

Def: A short exact sequence of
modules is $0 \rightarrow Z \xrightarrow{f} W \xrightarrow{g} Y \rightarrow 0$
where $\ker f = 0$, $\text{Im } f = \ker g$,
 $\text{Im } g = Y$. (Notice that g is an epimorphism
and f is a monomorphism.)

Consider the functor $F = \text{Hom}(X, -)$.

Fact: F is always a left-exact
covariant functor, meaning that

$0 \rightarrow \text{Hom}(X, Z) \xrightarrow{Ff} \text{Hom}(X, W) \xrightarrow{Fg} \text{Hom}(X, Y)$
is exact whenever $0 \rightarrow Z \xrightarrow{f} W \xrightarrow{g} Y \rightarrow 0$
is exact.

Prop/Def: The following properties of
left A -module X are equivalent:

1) Given an epimorphism (surjective)
 $g: W \rightarrow Y$ and a morphism $f: X \rightarrow Y$,
then there exists a morphism $h: X \rightarrow W$
s.t. $gh = f$.

2) Any epimorphism $g: W \rightarrow X$ splits, i.e.
there exists $h: X \rightarrow W$ s.t. $gh = \text{id}_X$.

3) There exists a left A -module Z
s.t. $X \oplus Z$ is a free A -module, i.e.
a direct sum of copies of A .

Rem: (3) \Rightarrow that any free A-module is projective.

- (3) 4) The functor $F = \text{Hom}(X, -)$ is exact, i.e. $0 \rightarrow Z \xrightarrow{f} W \xrightarrow{g} Y \rightarrow 0$ exact \Rightarrow
(*) $0 \rightarrow \text{Hom}(X, Z) \xrightarrow{Ff} \text{Hom}(X, W) \xrightarrow{Fg} \text{Hom}(X, Y) \rightarrow 0$ exact.

If any of these equivalent conditions are satisfied, we call X projective.

Before proving this prop, we need a Lemma:

Lemma/Def: The following three statements about a short exact sequence $0 \rightarrow W \xrightarrow{\alpha} V \xrightarrow{\beta} X \rightarrow 0$ are equivalent:

- 1) There exists $\gamma: V \rightarrow W$ s.t. $\gamma\alpha = \text{id}_W$
- 2) There exists $\delta: X \rightarrow V$ s.t. $\beta\delta = \text{id}_X$
- 3) $V \cong W \oplus X$.

This is called a split exact sequence.

Pf: Firstly, (3) \Rightarrow (1) and (2) by letting $\gamma: V \cong W \oplus X \rightarrow W$ & $\delta: V \rightarrow X$ be the natural projection maps.

(1) \Rightarrow (3): $v \in V$ is in $(\ker \gamma + \text{im } \alpha)$
since $v = (v - \alpha v) + \alpha v$ and
 $\gamma(v - \alpha v) = \gamma v - (\gamma \alpha)v = 0$.

Since $v \in (\ker \gamma) \cap (\text{im } \alpha) = \{0\}$
and $0 = \gamma v = (\gamma \alpha)v = \alpha v \Rightarrow v = 0$

it follows that $V \cong \ker \gamma \oplus \text{im } \alpha$.
 $\ker \beta = \text{im } \alpha$ by exactness, and $V \xrightarrow{\beta} X \rightarrow 0$ implies that β is surjective.

④ Thus for any $x \in X$, there exists $v \in V$ s.t.
 $x = Bv = B(k + \alpha w) = BK$ for
 $k \in \ker \delta, w \in W$

If $BK = 0$, then $K \in \ker B = \text{Im } \alpha$
and $K \in \ker \delta \cap \text{Im } \alpha \Rightarrow k = 0$.

\Rightarrow The restriction $B: \ker \delta \rightarrow X$
is an isomorphism.

Similarly, the exactness of $0 \xrightarrow{\alpha} W \xrightarrow{\beta} V$
implies that α is injective \Rightarrow

$W \cong \text{Im } \alpha$. Thus, the above direct
sum $\Rightarrow (3)$. (2) \Rightarrow (3) is analogous. \square

Now let us prove the Prop

(1) \Rightarrow (2) Let $X = Y, f = \text{id}_X$.

(2) \Rightarrow (3) Let $X \cong W/I$ where
 W is free, $g: W \xrightarrow{\cong} W/I$

and $Z = \ker g$.

We then have the short exact sequence

$$0 \rightarrow Z \hookrightarrow W \xrightarrow{g} X \rightarrow 0$$

$\cong_{\ker g}$ $\cong_{W/I}$

and epimorphism $g \circ h = \text{id}_X \Rightarrow$ sequence splits,
and we have $h: X \rightarrow W$ s.t. $gh = \text{id}_X$.

By splitting Lemma $\Rightarrow X \oplus Z \cong W$,
which is free.

⑤ (3) \Rightarrow (4) IF W is a free A -mod

$$\text{Hom}_A(W, M) \cong M \text{ for any } M.$$

Thus we have the exact sequence,

$$0 \rightarrow \text{Hom}_A(X \oplus Z, Z) \rightarrow \text{Hom}_A(X \oplus Z, W) \rightarrow \text{Hom}_A(X \oplus Z, Y) \rightarrow 0$$

which implies the two decompositions, including (*) are also exact.

(4) \Rightarrow (1) Since the sequence is exact, we have an induced surjection

$$\text{Hom}(X, W) \rightarrow \text{Hom}(X, Y)$$

$$h \mapsto gh = f \text{ whenever}$$

$g: W \rightarrow Y$ surjective.

Thus there always exists an h so that the following triangle commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ h \downarrow & \searrow & \\ W & \xrightarrow{g} & Y \rightarrow 0 \end{array} \quad \bullet \quad \boxed{\begin{array}{l} \text{General category} \\ \text{theoretic defn of} \\ \text{projective objects} \end{array}}$$

Def: An object/ A -module is called injective if given any monomorphism $g: W \rightarrow Y$, and any morphism $f: W \rightarrow X$, there exists a morphism $h: Y \rightarrow X$ such that $hg = f$.

$$\begin{array}{ccccc} 0 & \rightarrow & W & \xrightarrow{g} & Y \\ & & \boxed{\begin{array}{l} \text{Dual} \\ \text{Notion to} \\ \text{projectives} \end{array}} & \swarrow f & \downarrow h \\ & & & X & \end{array}$$

Notice that
arrows reversed, and
 W, Y switched

⑥ Can also think of X injective
 iff $\text{Hom}(-, X)$ is right exact
 iff monom. g induces surjective
 map $\text{Hom}(Y, X) \rightarrow \text{Hom}(W, X)$
 $h \mapsto hg = f$.

Our goal now is to describe projective and injective representations of quivers.

Def: For quiver Q and $x \in Q_0$, let
 P_x denote the rep corresp. to the
left KQ -module \underline{KQex} . For $y \in Q_0$,

$P_x(y) \hookrightarrow e_y KQex \hookrightarrow [x, y]$, the
vector space of all paths $x \rightarrow y$.
(This includes $e_x (= y=x)$)

$P_x(a)$ is the induced map $[x, sa] \rightarrow [x, ta]$
 $p \mapsto ap$.

Lemma (Yoneda): For any quiver rep
 V , the map $\text{Hom}_Q(P_i, V) \rightarrow V(i)$
 $\phi \mapsto \phi_i(e_i)$
is an isomorphism. [Analogous result
for injectives]

Pf: For $\phi \in \text{Hom}_Q(P_i, V)$, ϕ_i is
the restriction $P_i(i) \rightarrow V(i)$.
As noted above, $e_i \in P_i(i)$ and so
 $\phi_i(e_i) \in V(i)$ is well-defined.

We now show that the morphism ϕ is
completely determined by our choice of $\phi_i(e_i)$.

For $j \in Q_0$ and w a path in $[i, j]$,
 $\phi_j(w) = \phi_j(we_i) \stackrel{\text{e } ejkQet}{=} v(w)\phi_i(e_i)$

⑦ since the squares/rectangles

$$\begin{array}{ccc} P_i(i) & \xrightarrow{m \mapsto a_1 a_2 \dots a_n m} & P_i(j) \\ \phi_i \downarrow & & \downarrow \phi_j \\ V(i) & \xrightarrow{V(w) = V(a_1) \dots V(a_n)} & V(j) \end{array}$$

must commute for path $w = a_1 a_2 \dots a_n$.

Conversely, for $v \in V(i)$, let

$\phi_j(w)$ be defined as $V(w) \times$ for
any $j \in Q_0$ and path $V(a_1) \dots V(a_n) w \in [i, j]$.

This completely determines morphism ϕ .

Lemma IF Q has no oriented cycles,
then $\{P_i \mid i \in Q_0\}$ is a
complete set of pairwise non-isomorphic
representatives, each of which are indecomposable
and projective.

Pf: Firstly, $\text{End}_Q(P_i) \cong P_i(i) = [i, i]$
by Yoneda's Lemma, and since Q
has no oriented cycles, $[i, i] \cong k e_i \cong k$

By Fitting's Lemma $k \cong \text{End}_Q(P_i)$
local $\Rightarrow P_i$ indecomposable.

[Note: In class, we only proved the other
direction of Fitting's Lemma.]

P_i projective by (3) since if we let
 $v = P_1 \oplus \dots \oplus \widehat{P_i} \oplus \dots \oplus P_n$, $P_i \oplus v = kQ$, which is free.

$$Q_0 = \{1, 2, \dots, n\}$$

(8)

Now let V be an indecomp. $\text{Pr}(Q)$ -rep.

Let $P = P_1^{\dim_{\mathbb{K}} V(1)} \oplus \dots \oplus P_n^{\dim_{\mathbb{K}} V(n)}$ where $\dim_{\mathbb{K}} V(i) = d_i$ and we note that $\text{Hom}(P, V)$ contains the surjective morphism $\phi_Q : P \rightarrow V$ defined by picking bases $\{v_j^{(i)}\}_{j=1}^{d_i}$ for each $V(i)$ and setting $\phi_Q(e_i) = v_j^{(i)}$.

$$\phi_{V,j} : P_i \rightarrow V \text{ by } \phi_{V,j}(e_i) = v_j^{(i)}.$$

However V is projective so by (z) there exists a morphism $g : V \rightarrow P$ s.t. $\phi_Q \circ g = \text{id}_V$. Using similar tech. to spl. Lemma

$$P \cong \text{Ker}(g\phi) \oplus \text{Im}(g\phi) \cong V \text{ and}$$

by Krull-Schmidt, $V \cong P_i$ for some i .

If $i \neq j$, $[i, j]$ and $[j, i]$ cannot both be nonzero as Q has no oriented cycles.

$$\Rightarrow \text{Hom}_Q(P_i, P_j) = 0 \text{ or } \text{Hom}_Q(P_j, P_i) = 0$$

Either way, $P_i \not\cong P_j$ when $i \neq j$. \square

Remark: By similar results and the fact that injectives are defined dually to projectives,

we define (for $i \in Q_0$) I_i 's to be the quiver reps (for $j \in Q_0$)

$I_i(j) = [j, i]$, vector space of paths from j to i in Q
(can also be thought of as projectives in the opposite category, i.e. opposite quiver.)

$I_i(a)$'s analogously induced.

⑨ Lemma: The set of I_i 's are a complete set of pairwise non-isomorphic reps, each of which are indecomposable and injective.

Example $Q = \begin{matrix} & a \\ 1 & \xrightarrow{\quad} & 2 & b \\ & b \end{matrix}$

Projectives are

$$P_1 : \begin{matrix} Ke_1 & \xrightarrow{\cdot a} & Ka & \xrightarrow{\cdot b} & Kba \\ [1,1] & & [1,2] & & [1,3] \end{matrix}$$

$$P_2 : \begin{matrix} O & \xrightarrow{\circ} & Ke_2 & \xrightarrow{\cdot b} & Kb \\ [2,1] & & [2,2] & & [2,3] \end{matrix}$$

$$P_3 : \begin{matrix} O & \xrightarrow{\circ} & O & \xrightarrow{\circ} & Ke_3 \\ [3,1] & & [3,2] & & [3,3] \end{matrix}$$

$\cong E_3$

$$\begin{aligned} \text{Notice } P_1 &\cong K \xrightarrow{\downarrow} K \xrightarrow{\downarrow} K = I_{13} \\ P_2 &\cong O \xrightarrow{\circ} K \xrightarrow{\downarrow} K = I_{23} \end{aligned}$$

Injectives are

$$I_1 \stackrel{\cong E_1}{\sim} \begin{matrix} Ke_1 & \rightarrow & O & \rightarrow & O \\ [1,1] & & [2,1] & & [3,1] \end{matrix}$$

$$I_2 : \begin{matrix} Ka & \rightarrow & Ke_2 & \rightarrow & O \\ [1,2] & & [2,2] & & [3,2] \end{matrix}$$

$$I_3 : \begin{matrix} K(ba) & \rightarrow & Kb & \rightarrow & Ke_3 \\ [1,3] & & [2,3] & & [3,3] \end{matrix}$$

⑩ After spring break: We will define a notion of almost split exact sequences and use this to describe cluster mutation in terms of categorical language.

Will also describe reflection functors and prove Gabriel's Theorem.

Teaser: Knitting algorithm for

$$A_3 \quad \begin{matrix} & \circ & \circ & \circ \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{matrix}$$

$$\begin{array}{ccc} \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] P_3 = E_3 & \xrightarrow{\quad} & \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] E_2 \\ \downarrow & & \downarrow \\ \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] P_2 & \xrightarrow{\quad} & I_2 \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right] \\ \downarrow & & \downarrow \\ \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] P_1 = I_3 & & \end{array}$$

$$P_1 = K \overset{1}{\rightarrow} K \overset{1}{\rightarrow} K = I_3$$

$$P_2 = O \rightarrow K \overset{1}{\rightarrow} K$$

$$P_3 = O \rightarrow O \rightarrow K = E_3$$

$$I_1 = K \rightarrow O \rightarrow O = E_1$$

$$I_2 = K \overset{1}{\rightarrow} K \rightarrow O$$

$$E_2 = O \rightarrow K \rightarrow O$$

dim
vectors
satisfy
"tropical"
diamond
condition

a
b
c
d
 $a+d = b+c$