

Lecture 16: BGP Reflection Functors (3-21-11)

Gregg Musiker Math 8680 §PF of Gabriel's Thm I

Note Title

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- ① For today's class, we follow sections 5.4-5.7 of "Introduction to Rep'n theory" by Etingof, Golberg, Hensel, Liu, Schwendner, Vaintroub, and Yudovina.

Thm (Gabriel) Let Q be a quiver that is oriented Dynkin Diagram Γ of type $A_n, D_n, E_6, E_7, \text{ or } E_8$.

Then (1) Q has finitely many indecomp. representations (up to isomorphism)

(2) For each positive root α of the root system corresponding to Γ is the dim vector of a unique (up to isom.) indecomp. rep.

(3) The correspondence in (2) goes both directions, i.e. if V is an indecomp. rep of quiver Q of type Γ , then dim V is a pos. root of Γ 's root system.

(4) If Q is not as described in (1), then Q is not of finite rep type.

Some definitions before we start the proof:

Def: Let Q be any quiver and $i \in Q_0$ be a sink. Let V be a rep of Q .

We define the BGP-reflection functor

$$C_i^+ : \text{rep}_K Q \rightarrow \text{rep}_K \overline{Q}_i \quad \left[\overline{Q}_i := Q \text{ w/ all arrows incident to } i \text{ reversed} \right]$$

② by $C_i^+(V)_K = V_K$ if $K \neq \bar{i}$,
 $C_i^+(V)_i = \text{Ker} \left(\varphi: \bigoplus_{\bar{j} \rightarrow \bar{i}} V_j \rightarrow V_i \right)$

C_i^+ also leaves all linear maps alone except for those incident to \bar{i} .

For $K \xrightarrow{a} \bar{i}$, an arrow of Q , we have arrow $\bar{i} \xrightarrow{\bar{a}} \bar{K}$ in \bar{Q}_i as

\bar{i} is now a source instead of a sink.

$$C_i^+(\bar{a}): \text{Ker } \varphi \hookrightarrow \bigoplus_{\substack{\bar{j} \rightarrow \bar{i} \\ \text{in } \bar{Q}}} V_j \twoheadrightarrow V_K$$

the composition of the natural inclusion and projection maps.

We define the reflection functor also when $\bar{i} \in Q_0$ is a source:

$$C_i^-(V)_K = V_K \text{ if } K \neq \bar{i}$$

$$C_i^-(V)_i = \text{coker} \left(\psi: V_i \rightarrow \bigoplus_{\substack{\bar{i} \rightarrow \bar{j} \\ \text{in } \bar{Q}}} V_j \right) \\ = \bigoplus_{\substack{\bar{i} \rightarrow \bar{j} \\ \text{in } \bar{Q}}} V_j / \text{Im } \psi.$$

C_i^- also leaves arrows alone except

$$C_i^-(\bar{a}): V_K \hookrightarrow \bigoplus_{\substack{\bar{i} \rightarrow \bar{j} \\ \text{in } \bar{Q}}} V_j \twoheadrightarrow \text{coker } \psi.$$

③ Examples: Let $Q = \begin{array}{c} 3 \\ \bullet \end{array} \xleftarrow{b} \begin{array}{c} 2 \\ \bullet \end{array} \xleftarrow{a} \begin{array}{c} 1 \\ \bullet \end{array}$
 and $V = P_1 = \begin{array}{c} \bullet \\ \downarrow \\ K \end{array} \xleftarrow{1} \begin{array}{c} \bullet \\ \downarrow \\ K \end{array} \xleftarrow{1} \begin{array}{c} \bullet \\ \downarrow \\ K \end{array}$

$C_3^+(V)$ is the \overline{Q}_3 -rep V'
 $\cong \begin{array}{c} 3 \\ \bullet \end{array} \xrightarrow{b} \begin{array}{c} 2 \\ \bullet \end{array} \xleftarrow{a} \begin{array}{c} 1 \\ \bullet \end{array}$

$V'(1) = K, V'(2) = K, V'(a) = 1 \text{ map},$

$V'(3) = \text{Ker} \left(\begin{array}{c} K \\ \bullet \\ 2 \end{array} \xrightarrow{1} \begin{array}{c} K \\ \bullet \\ 3 \end{array} \right) = 0,$

$V'(b) = \text{composition} = 0 \text{ map}$

$$\begin{array}{c} 0 \hookrightarrow K \twoheadrightarrow K \\ \bullet \quad \bullet \\ \downarrow \quad \downarrow \\ 2 \quad 2 \end{array}$$

$\text{Ker} \left(\begin{array}{c} K \xrightarrow{1} K \\ \bullet \quad \bullet \\ 2 \quad 3 \end{array} \right)$

$\Rightarrow V' \cong \begin{array}{c} 0 \\ \bullet \\ 3 \end{array} \xrightarrow{0} \begin{array}{c} K \\ \bullet \\ 2 \end{array} \xleftarrow{1} \begin{array}{c} K \\ \bullet \\ 1 \end{array}$

$C_1^-(V)$ is the \overline{Q}_1 -rep V''
 $\cong \begin{array}{c} 3 \\ \bullet \end{array} \xleftarrow{b} \begin{array}{c} 2 \\ \bullet \end{array} \xrightarrow{a} \begin{array}{c} 1 \\ \bullet \end{array}$

$V''(2) = K, V''(3) = K, V''(b) = 1 \text{ map},$

$V''(1) = K / \text{Im} \left(\begin{array}{c} K \xrightarrow{1} K \\ \bullet \quad \bullet \\ 1 \quad 2 \end{array} \right) \cong 0,$

$V''(a) = \text{composition} = 0 \text{ map}$

$$\begin{array}{c} K \hookrightarrow K \twoheadrightarrow K / \text{Im}(K \rightarrow K) \\ \bullet \quad \bullet \quad \bullet \\ \downarrow \quad \downarrow \quad \downarrow \\ 2 \quad 2 \quad 2 \end{array}$$

$\Rightarrow V'' \cong \begin{array}{c} K \\ \bullet \\ 3 \end{array} \xleftarrow{1} \begin{array}{c} K \\ \bullet \\ 2 \end{array} \xrightarrow{0} \begin{array}{c} 0 \\ \bullet \\ 1 \end{array}$

④ Consider rep W of $Q_3 = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \end{array}$ $\begin{array}{c} K \\ \xrightarrow{2} \\ K \\ \xrightarrow{1} \\ K \end{array}$

$W' = C_2^+(W)$ is a rep of $Q_2' = Q_1 = \begin{array}{c} \bullet \\ \leftarrow \\ \bullet \\ \rightarrow \\ \bullet \end{array}$ $\begin{array}{c} K \\ \xrightarrow{2} \\ K \\ \xrightarrow{1} \\ K \end{array}$

$$W'(2) = \text{Ker}(\varphi: K^2 \rightarrow K)$$

$$W(1) \oplus W(3) \rightarrow W(2)$$

$$(w_1, w_3) \mapsto w_1 + w_3$$

$$= \{(w_1, -w_1) : w_1 \in K\} \cong K$$

$$W'(\bar{b}) : \text{Ker } \varphi \hookrightarrow W(1) \oplus W(3) \twoheadrightarrow W(3)$$

$$(-w_3, w_3) \longmapsto w_3$$

$$\cong \text{identity map } K \rightarrow K$$

Same for $W'(\bar{a}) \cong \mathbb{1}_{\text{map}} \Rightarrow W' \cong \begin{array}{c} \bullet \\ \leftarrow \\ K \\ \leftarrow \\ \bullet \end{array} \begin{array}{c} K \\ \xrightarrow{1} \\ K \\ \xrightarrow{1} \\ K \end{array}$

$W'' = C_2^-(W')$ is a rep of Q'

$$W''(2) = (W'(1) \oplus W'(3)) / \text{Im}(\psi: W'(2) \rightarrow W'(1) \oplus W'(3))$$

$$= \{(w, w') : \begin{array}{c} K \rightarrow K^2 \\ w \mapsto (w, w) \end{array}\}$$

$$\{(w, w) \sim 0\} \cong \{(\bar{w}, 0) : w \in K\} \cong K$$

$$W''(\bar{b}) : W'(3) \hookrightarrow W'(1) \oplus W'(3) \twoheadrightarrow \text{coker } \psi$$

$$w \mapsto (0, w) \mapsto (\bar{0}, w)$$

Same for $W''(\bar{a}) \Rightarrow W'' \cong W \begin{array}{c} \bullet \\ \xrightarrow{1} \\ K \\ \xrightarrow{1} \\ \bullet \end{array} \begin{array}{c} K \\ \leftarrow \\ K \\ \leftarrow \\ K \end{array}$

⑤ Just saw an example where $C_2^- C_2^+ W \cong W$.
 More on this phenomenon in a minute.

Prop 1: Let Q be a quiver and V an indecomposable rep of Q .

1) Let $i \in Q_0$ be a sink. Then either $V \cong E_i$ (simple rep) OR the map $\varphi: \bigoplus_{j \rightarrow i} V_j \rightarrow V_i$ is surjective.

2) Let $i \in Q_0$ be a source. Then either $V \cong E_i$ OR the map $\psi: V_i \rightarrow \bigoplus_{i \rightarrow j} V_j$ is injective.

Pf: We prove (1) since the proof of (2) is analogous.

If φ is not surjective let W denote a vec. sp. giving a complement of $\text{Im } \varphi$.

$\Rightarrow V \cong \begin{array}{ccc} & W & \\ \nearrow & \uparrow & \nwarrow \\ 0 & 0 & 0 \\ & \vdots & \\ & 0 & \end{array} \oplus V'$ where V' is a Q rep.

However, since V was assumed to be indecomposable, $W=0$ or V' is the 0 rep.

If V' is the 0 rep, then V only indecomp if $W \cong K \Rightarrow V \cong E_i$

If $W=0$, then φ is surjective. \square

⑥ Prop 2: Let Q be a quiver, V be any Q -rep.

1) If i is a sink and $\varphi: \bigoplus_{j \rightarrow i} V_j \rightarrow V_i$ is surjective, then $c_i^- c_i^+ V = V$, again a Q -rep.

2) If i is a source, $\psi: V_i \rightarrow \bigoplus_{i \rightarrow j} V_j$ is injective, then $c_i^+ c_i^- V = V$.

PF: We prove (2) this time. Again, the proof of (1) is analogous.

Let $\psi: V_i \rightarrow \bigoplus_{\substack{i \rightarrow j \\ \text{in } Q}} V_j$ be injective.

Then $V' := c_i^- V$ satisfies

$$V'(i) = \bigoplus_{\substack{i \rightarrow j \\ \text{in } Q}} V_j / (\text{Im } \psi).$$

We then apply c_i^+ to V' and get V''

$$\begin{aligned} V''(i) &= \ker\left(\varphi: V'(i) \rightarrow \bigoplus_{\substack{i \rightarrow j \\ \text{in } Q}} V_j\right) \\ &= \ker\left(\varphi: \bigoplus_{\substack{i \rightarrow j \\ \text{in } Q}} V_j / \text{Im } \psi \rightarrow \bigoplus_{\substack{i \rightarrow j \\ \text{in } Q}} V_j\right) \end{aligned}$$

$\cong \text{Im } \psi$ by Isom Theorem

$\cong V_i$ since ψ is injective \square

⑦ Prop 3: Let Q be a quiver and V an indecomposable rep of Q . Then (when they are defined)

$C_i^+ V$ and $C_i^- V$ are either 0 or again indecomposable.

Pf: Assume i is a sink. By Prop 1, indecomp. V is either $\cong E_i$ or the map $\varphi: \bigoplus_{j \rightarrow i} V_j \rightarrow V_i$ is surjective.

Notice that $C_i^+ E_i = 0$ as $\varphi: \bigoplus_{j \rightarrow i} V_j \cong 0 \rightarrow V_i \cong k$ has $\ker \cong 0$.

Thus assume that φ is surjective.

If $V' := C_i^+ V$ is decomposable, i.e.

$V' = U \oplus W$ (with $U, W \neq 0$), then

$$C_i^- V' = C_i^- U \oplus C_i^- W$$

$$\parallel$$

$$C_i^- C_i^+ V$$

$$\parallel \leftarrow \text{by Prop 2 (1)}$$

$$V$$

We claim that neither $C_i^- U$ nor $C_i^- W$ can be $\cong 0$ because

$V' = C_i^+ V$ is injective at i since the maps are canonical projectives and the decomposition into U and W is compatible with this embedding.

⑧ Thus u, w also injective at i , and by Prop 2 (2) satisfy

$$c_i^+ c_i^- w = w, c_i^+ c_i^- u = u$$

Since $c_i^+ 0 = 0 \Rightarrow c_i^- w, c_i^- u \neq 0$

But V was assumed to be indecomp.,

so $V \cong c_i^- w \oplus c_i^- u$ is a contradiction,

$\Rightarrow V' = c_i^+ V$ was indecomposable.

As above, the proof that $c_i^- V$ is 0 or indecomposable follows analogously. \square

We now have recorded a number of properties of the BGP reflection functors. Before stating the last property that we will need for now, we need a quick description of root systems for generalized Cartan matrices, and the relation to quivers:

Let Q be a **loopless** quiver on n vertices.

We build an abstract root system with simple roots $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and reflections $s_i(\alpha) = \alpha - (\alpha, \alpha_i) \alpha_i$ where (\cdot, \cdot) is a symmetric bilinear form, called the Ringel or Euler form,

defined by $(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i=j \\ -(\# \text{ arrows in } Q \\ i \rightarrow j \text{ or } j \rightarrow i) & \text{if } i \neq j \end{cases}$

$\alpha = c_1 \alpha_1 + \dots + c_n \alpha_n$.
 More on this form in a later lecture

⑨ Prop 4: Let Q be a quiver and V , a representation of Q .

1) IF $i \in Q_0$ is a sink and V is surjective at i , then $\underline{\dim} C_i^+ V = s_i(\underline{\dim} V)$

where s_i denotes the i th simple reflection corresponding to quiver Q .

2) IF $i \in Q_0$ is a source and V is injective at i , then $\underline{\dim} C_i^- V = s_i(\underline{\dim} V)$.

Pf: We prove (2), noting that (1) follows similarly.

Let i be a source and $\psi: V_i \rightarrow \bigoplus_{i \rightarrow j \text{ in } Q} V_j$ be injective.

Let C_K denote coker ψ

$$\dim C_K = \sum_{i \rightarrow j} \dim V_j - \dim V_i$$

since ψ is injective.

Note: This sum is over all

Therefore $\dim(C_i^- V) - \dim V_i$ arrows incident to i .

$$= \sum_{i \rightarrow j} \dim V_j - 2 \dim V_i, \text{ and}$$

$\underline{\dim} C_i^- V$ and $\underline{\dim} V$ only can differ in their i th components. Thus

If we let $\underline{\dim} V = \alpha = c_1 \alpha_1 + \dots + c_n \alpha_n$

then $\underline{\dim} C_i^- V = c_1 \alpha_1 + \dots + c_i' \alpha_i + \dots + c_n \alpha_n$

$$\text{where } c_i' = \left(\sum_{i \rightarrow j} c_j \right) - c_i \Rightarrow$$

$$\textcircled{10} \quad c_i' = c_i - \left(2c_i - \sum_{i \rightarrow j} c_j \right)$$

Comparing with above, we see that

$$\begin{aligned} \underline{\dim} C_i^- V &= c_1 \alpha_1 + \dots + c_i' \alpha_i + \dots + c_n \alpha_n \\ &= s_i(c_1 \alpha_1 + \dots + c_n \alpha_n) \\ &= s_i(\underline{\dim} V). \quad \square \end{aligned}$$

We now know a number of properties about reflection functors including that they behave like simple reflections. We will see next week how we exploit this to finish the proof of Gabriel's Theorem.

Note that (\cdot, \cdot) being pos definite is crucial. On H.W 2, you describe for what quivers this condition is satisfied.

Next class, we take a segway and talk about the Caldero-Chapoton map and how reflection functors relate to cluster algebras.