

Lecture 17: Gabriel's Thm, Part II, and the Gregg Musiker 8680 Caldero-Chapoton Map (3-23-11)

Note Title

3/17/2011

① Recall from last class that we introduced the BGP reflection factors C_i^+ and C_i^- , and proved that they have the following properties:

Let Q be a quiver, and let V be an indecomposable rep. of Q .

1) IF $i \in Q_0$ is a sink and $V \subseteq E_{ij}$ then $C_i^+ V = 0$. If $V \not\subseteq E_{ij}$ then $C_i^+ V = sV$, another indecomposable s.t. $\dim sV = s_i (\dim V)$.

2) IF $i \in Q_0$ is a source, then the same statements hold, but with C_i^- 's replacing the C_i^+ 's.

E.g. $s_2 \xrightarrow{c} \alpha_1 + \alpha_2 + \alpha_3 \xrightarrow{s_1}$

$\alpha_1 + \alpha_2 \xleftarrow{c} \alpha_2 + \alpha_3$ Let $c = s_3 s_2 s_1$

$s_3 \xleftarrow{c} \alpha_1 \quad \alpha_2 \xleftarrow{c} \alpha_3 \quad \alpha_3 \xleftarrow{c} s_1$

Analogous action on indecomp reps. (see Pg. 3)

We now continue with secs 5.7-5.8 of Etingof et. al., and the proof of Gabriel's Theorem.

Let V be an indecomposable of Q .

If Q is acyclic, then we can label Q_0 as $\{1, 2, \dots, n\}$ s.t. $i < j$ IF there is a path from i to j . This means that 1 labels a source, and n labels a sink.

② Def: Let Q be a quiver and Γ be its underlying undirected graph.

Given a labeling $1, 2, \dots, n$ of the vertices of Q_0 and Γ , we let the corresponding Coxeter element c of Q be defined as $c = s_1 s_2 \cdots s_n$.

Lemma: Let $\alpha = c_i \alpha_i + \cdots + c_n \alpha_n \neq 0$

with $c_i \geq 0$ for all i . Assume that the form (\cdot, \cdot) defined above is positive definite, i.e. $(x, x) \geq 0$ with $(x, x) = 0 \Leftrightarrow x = 0$.

Then, there exists an $N \in \mathbb{N}$ s.t. $c^N \alpha$ has at least one negative coefficient.

Pf: We first claim that the s_i 's generate a finite group ($\Rightarrow (\cdot, \cdot)$ is positive definite). (See Sec. 6.4 of Humphrey's "Reflection Grps and Coxeter Gps".)

Sylvester's: A real symm. matrix M is pos def (\Leftrightarrow all principal minors, including M , have pos det).

Thus, c has finite order $\nmid c^M = 1$ for $M \in \mathbb{N}$.

c does not have 1 as an eigenvalue, because otherwise there would exist $v \neq 0$ s.t. $c v = v$

$$s_1 s_2 \cdots s_n v \stackrel{\text{if}}{=} s_2 \cdots s_n v = s_1 v$$

However, s_i only changes i -th coord, so we would get $s_1 v = v = s_2 \cdots s_n v$, and iterating this procedure, $s_i v = v \forall i \in Q_0$

③ $\Rightarrow (\alpha_{ij}, \alpha_j) = 0$ for all i, j ,
disagreeing w/ above def.

Thus can factor $(c^M - 1) = 0$ as

$$(1 + c + c^2 + \dots + c^{M-1})(c-1) = 0 \Rightarrow$$

For all $v \neq 0$, there exists $w = (c-1)v \neq 0$

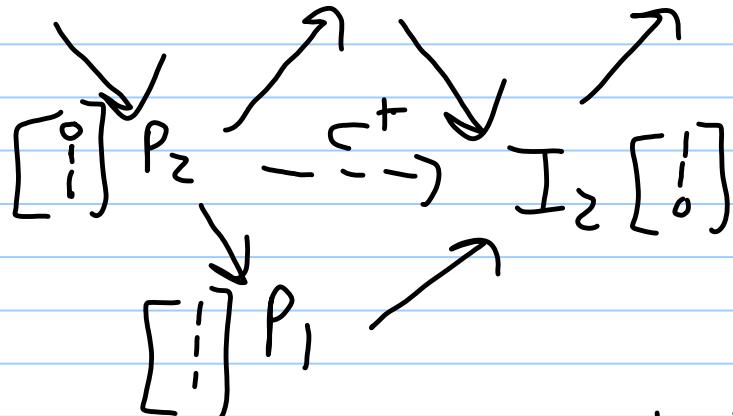
$$\text{s.t. } (1 + c + c^2 + \dots + c^{M-1})w = 0$$

$$\Rightarrow \text{operator } (1 + c + c^2 + \dots + c^{M-1}) = 0.$$

$$\text{Thus } \alpha + [c\alpha + c^2\alpha + \dots + c^{M-1}\alpha] = 0$$

where α has at least one positive coefficient \Rightarrow one of the $c^N\alpha$'s has at least one negative coeff,

$$P_3 = E_3 \xrightarrow{c^+} E_2 \xrightarrow{c^+} E_1$$



$$c^+ = c_3^+ c_2^+ c_1^+$$

$$c_1^+ \xrightarrow{\begin{matrix} 3 \\ 0 \end{matrix}} \xrightarrow{\begin{matrix} 2 \\ 0 \end{matrix}} \xrightarrow{\begin{matrix} 1 \\ 0 \end{matrix}} c_2^+ \xrightarrow{\begin{matrix} 3 \\ 0 \end{matrix}} \xleftarrow{\begin{matrix} 2 \\ 0 \end{matrix}} \xleftarrow{\begin{matrix} 1 \\ 0 \end{matrix}} c_3^+$$

(4)

We now consider a sequence of representations (of different quivers):

$$V = V^{(0)}, \underset{\parallel}{V}^{(1)}, \underset{\parallel}{V}^{(2)}, \dots$$

$$C_n^+ V \quad C_{n-1}^+ C_n^+ V$$

Notice that because of our labeling, vertex i is a sink in the quiver associated to rep $C_{i+1}^+ C_{i+2}^+ \dots C_n^+ V$ for $1 \leq i \leq n$.

Furthermore, $V^{(n)} = C_1^+ C_2^+ \dots C_n^+ V$ is actually a rep of Q again as every arrow has now been reversed exactly twice.

Example $Q = \begin{array}{c} 1 \\ \rightarrow \\ 2 \\ \leftarrow \\ 3 \end{array}$

$$Q^{(1)}: \begin{array}{c} 1 \\ \rightarrow \\ 2 \\ \leftarrow \\ 3 \end{array}$$

$$Q^{(2)}: \begin{array}{c} 1 \\ \leftarrow \\ 2 \\ \rightarrow \\ 3 \end{array}$$

$$Q^{(3)}: \begin{array}{c} 1 \\ \rightarrow \\ ? \\ \rightarrow \\ 3 \end{array} = Q$$

Thus we can define $V^{(n+1)} = C_n^+ V^{(n)}$ and continue the sequence indefinitely.

$$\text{Let } C^+ = C_1^+ C_2^+ \dots C_n^+.$$

The sequence $V_j C^+ V = V^{(n)} (C^+)^2 V = V^{(2n)}$ consist only of reps of Q .

We thus can compare dimension vectors.

Prop: Let V be an indecomposable rep. of Q , where we assume Q is of type A_n, D_n, E_6, E_7 , or E_8 .

⑤ Then there exists N s.t.

$$V^{(N)} \cong E_p \text{ for some } p.$$

Pf:

We first show that

$$c_k^+ c_{k+1}^+ \cdots c_n^+ (c^+)^M V \cong E_{k-1} \text{ for some } k \text{ and } M.$$

Assume otherwise. Then for any k, M

$$\begin{aligned} \dim(c_k^+ c_{k+1}^+ \cdots c_n^+ (c^+)^M V) \\ = s_k s_{k+1} \cdots s_n \underset{\dim}{\underline{\dim}} V. \end{aligned}$$

Since $\dim V$ can be written as $c_1 \alpha_1 + \cdots + c_n \alpha_n$ with $c_i \geq 0$, it follows that for some M , $c^M \dim V$ has a negative coeff in its expansion. Contradicts the fact that \dim is a dimension vector.

Thus, we must have $V^{(nM+k)} \cong E_{k-1}$ so that $c_{k-1}^+ V^{(nM+k)} = 0$. \blacksquare

Corl Let Q be a quiver of type $\Gamma = \text{An, D}_n \text{ or } E_6, E_7, E_8$ and V be an indecomp. Q rep. Then $\dim V$ is a positive root of Γ 's root system.

Pf: By the Proposition there exists an M and K s.t. $c_{k+1}^+ c_{k+2}^+ \cdots c_n^+ (c^+)^M V \cong E_K$.

In fact K and M must be the

⑥ only choices of indexes $N_j p$ s.t.

$$c_{p+1}^+ c_{p+2}^+ \dots c_n^+ (c^+)^N V \cong E_p \text{ since}$$

$c_p^+ E_p = 0$ and the sequence then would terminate.

Thus, by earlier Propositions, all the c_p^+ 's act like s_p on the dim. vectors.

$$\begin{aligned} \Rightarrow \alpha_k &= \dim E_p \\ &= s_{k+1} s_{k+2} \dots s_n c^M \dim V \\ \Rightarrow \dim V &= (c^{-1})^M s_n \dots s_{k+2} s_{k+1} \alpha_k \end{aligned}$$

and applying simple reflections to the simple root α_k yields a root in the root system. It is nec. positive since its coordinates are dimensions.

Cor 2: If V and V' are two indecomposables which both have dimension vector $\longleftrightarrow \alpha$, a pos root, then $V \cong V'$.

Pf: By the above, there exists $k \in M$ s.t.

$$c_{k+1}^+ c_{k+2}^+ \dots c_n^+ (c^+)^M V \supset$$

$$c_{k+1}^+ c_{k+2}^+ \dots c_n^+ (c^+)^M V' \text{ both } \cong E_k.$$

⑦ we then let $c^- = c_n^- c_{n-1}^- \dots c_1^-$
and note that in the sequence

$(c^-)^M c_n^- \dots c_{k+2}^- c_{k+1}^-)$ the functor

c_p^- always acts on a source
(by symmetry w/ c_p^+ 's only acting at sinks)

$\Rightarrow (c^-)^M c_n^- \dots c_{k+2}^- c_{k+1}^- c_{k+1}^+ c_{k+2}^+ \dots c_n^+ (c^+)^M V$

$\& (c^-)^M c_n^- \dots c_{k+2}^- c_{k+1}^- c_{k+1}^+ c_{k+2}^+ \dots c_n^+ (c^+)^M V'$

are isomorphic.

Lastly, pairing off functors from the middle (since we do not obtain E_p as a result for smaller indices). These two strings are just $\cong V \& \cong V'$, respectively, as $c_p^- c_p^+ V \cong V$ if $V \not\cong E_p$.

We thus have proven a bijection (for Q of type A_n, D_n, E_6, E_7 , or E_8)

{positive roots of
assoc. root system} \longleftrightarrow {isomorphism classes
of indecomp. Q reps}

$\alpha \longleftrightarrow V$ with dim $V = \alpha$

Further for such representations, they are the only rep. up to isomorphism with the same dimension vector.

Such reps are known as rigid.

(8)

Note: An alternative definition of rigid, along w/ equivalence to the above definition will be given later.

Such a bijection in fact holds in more generality:

Thm: Let Q be a finite quiver w/o oriented cycles (on vertex set $\{1, 2, \dots, n\}$).

Then we have a bijection

$$\left\{ \begin{array}{l} \text{real positive} \\ \text{roots in "root system"} \\ \text{corresp. to } Q \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{isom. classes} \\ \text{of indecomp.} \\ \text{rigid } Q\text{-reps} \end{array} \right\}$$

Pf: Follows from above arguments.

— If Q is a general acyclic quiver, $\langle \cdot, \cdot \rangle$ is not nec. positive definite, and $W \langle S_1, \dots, S_n \rangle$ do not nec. generate a finite group (as we will see.)

Thus the Lemma from last time, i.e. (3) from today fails, and we cannot assume that

all positive roots are reachable from a simple one via by W .

But recall: those that are reachable are known as real roots.

We will prove this more general Thm, which is a corollary of a result known as Kac's Theorem, next week.

⑨ While the full proof of Gabriel's Thm and Kac's Theorem still awaits, we remind ourselves that we had also previously seen, from Fomin-Zelevinsky's Finite Type Classification that for any acyclic cluster algebra of type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ or G_2 , we have a bijection

$$\left\{ \begin{array}{l} \text{positive roots of} \\ \text{assoc. root system} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{non-initial} \\ \text{cluster variables in} \\ \text{assoc. cluster alg.} \end{array} \right\}$$

$$\alpha \xleftrightarrow{\sim} X_\alpha \text{ with denominator } x_1^{c_1} x_2^{c_2} \dots x_n^{c_n}$$

$$c_1\alpha_1 + \dots + c_n\alpha_n$$

Note that A_n, D_n, E_6, E_7, E_8 are the skew-symmetric (simply-laced) cluster algebras of finite type.

We now describe Caldero-Chapoton's formula relating an indecomposable quiver rep directly with a cluster variable.

We begin by defining the Quiver Grassmannian:

Def: Let Q be a quiver, V a rep of Q , and $\vec{e} \in \mathbb{N}^{Q_0}$ such that $0 \leq e_i \leq \dim V_i$ for each $i \in Q_0$.

$\text{Gr}_{\vec{e}}(V) =$ the variety whose points parametrize subspaces $U_i \subseteq V_i$ $\dim U_i = e_i$ for each $i \in Q_0$, and the U_i 's glue together to form a subrep $U \subseteq V$.

$$\text{Gr}_{\vec{e}}(V) \hookrightarrow \prod_{i=1}^n \text{Gr}_{e_i}(V_i)$$

where the RHS consists of n -tuples of points in ordinary Grassmannians (each of which has the structure of a projective variety).

(10) Example: $Q = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \xrightarrow{\quad} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$
 $V = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \xrightarrow{\mathbb{K}^2} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \xrightarrow{\mathbb{K}^2}$

$$\text{Gr}_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}(V) = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right\}, \quad \text{Gr}_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}(V) \cong \text{Gr}_1(V_1), \\ \cong \mathbb{P}^1$$

$$\text{Gr}_{\begin{bmatrix} ? \\ 1 \end{bmatrix}}(V) = \emptyset, \quad \text{Gr}_{\begin{bmatrix} 1 \\ ? \end{bmatrix}}(V) \cong \text{Gr}_1(V_1), \\ \cong \mathbb{P}^1$$

$$\text{Gr}_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}(V) = \emptyset, \quad \text{Gr}_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(V) \cong \text{Gr}_1(V_2) \\ \cong \mathbb{P}^1$$

$$\text{Gr}_{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}(V) = \left\{ \begin{array}{c} 0 \\ \bullet \\ \bullet \end{array} \xrightarrow{\mathbb{K}^2} \right\}, \quad \text{Gr}_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}(V) \cong \left\{ \begin{array}{c} 0 \\ \bullet \\ \bullet \end{array} \xrightarrow{\text{both single pts}} \right\}$$

Note: $\text{Gr}_{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}(V) \cong \text{Gr}_1(V_{(1)}) \cong \mathbb{P}^1$ since we choose any 1-dim subspace $U_1 \subseteq V_1 \cong \mathbb{K}^2$ and $\text{Im}(\mathcal{I}_2|_{U_1}) \subseteq V_2 \cong \mathbb{K}^2$ automatic.

1-dim subspace of \mathbb{K}^2 looks like $\langle \begin{bmatrix} 1 \\ y \end{bmatrix} \rangle$ for $y \in \mathbb{K}$, or $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$, hence $\cong \mathbb{P}^1$.

For $\text{Gr}_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}(V)$, again choose any $U_1 \subseteq V_1$ but then $U_2 \subseteq V_2$ forced to be $\text{Im}(\mathcal{I}_2|_{U_1}) \cong \mathbb{K}$.

Finally, for $\text{Gr}_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}(V)$, we pick any 1-dim subspace $U_2 \subseteq V_2$ and $\text{Im}(\mathcal{I}_2|_0) = \emptyset \subseteq U_2$ satisfied.

In general, the conditions needed so that the U_i 's glue together into a subrep is a closed condition $\Rightarrow \text{Gr}_p(V)$ is a projective subvariety of $\prod_{i=1}^n \text{Gr}_{e_i}(V_i)$.

(11)

The Caldero-Chapoton formula involves calculating the Euler characteristic χ of $\text{Gr}_{\vec{e}}(V)$.

Recall for a projective variety

$$\chi(\bullet) = 1, \quad \chi(\mathbb{P}^n) = n+1, \text{ and}$$

$$\text{if } V = V_1 \cup V_2 \cup \dots \cup V_m, \quad \chi(V) = \chi(V_1) + \dots + \chi(V_m).$$

So in the above example,

$$\chi(\text{Gr}_{\vec{e}}(v)) = \begin{cases} 2 & \text{if } \vec{e} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ 1 & \text{if } \vec{e} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ 0 & \text{if } \vec{e} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

Def: The Caldero-Chapoton Formula of a quiver rep (also known as cluster character) is

$$CC(v) = \frac{1}{x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}} \sum_{\substack{0 \leq e_i \leq d_i \\ \text{for } 1 \leq i \leq n}} \chi(\text{Gr}_{\vec{e}}(v)) \prod_{i=1}^n x_i^{c_i}$$

$$\text{where } c_i = \sum_{j \neq i} e_j + \sum_{i \rightarrow j} (d_j - e_j).$$

e.g. $Q = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \rightarrow \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$, $V = k^2 \xrightarrow{\text{Id}} k^2$,

$$c_1 = d_2 - e_2 = 2 - e_2, \quad c_2 = e_1$$

$$\Rightarrow CC(v) = \frac{(x_1^0 x_2^2 + x_1^0 x_2^0 + x_1^2 x_2^0 + 2 x_1^0 x_2^1 + 2 x_1^1 x_2^1 + 2 x_1^1 x_2^0)}{x_1^2 x_2^2}$$

$$= \left(\frac{x_2 + 1 + x_1}{x_1 x_2} \right)^2.$$

12

Thm (Caldero-Chapoton): Let Q be a quiver of type A_n, D_n, E_6, E_7 , or E_8 . Let V be an indecomposable rep. of Q .

Then $CC(V) = X_V$ the cluster variable s.t. $\text{denom}(x_v) = \dim V$, and this gives a bijection between indecomposables and the non-initial cluster variables.

Example: $Q = \begin{matrix} & \bullet \\ 1 & \rightarrow & \bullet \\ & \downarrow & \\ & \bullet & \end{matrix} \rightarrow \begin{matrix} & \bullet \\ 2 & \rightarrow & \bullet \\ & \downarrow & \\ & \bullet & \end{matrix} \rightarrow \begin{matrix} & \bullet \\ 3 & \rightarrow & \bullet \\ & \downarrow & \\ & \bullet & \end{matrix}$

Let us consider all the indecomposables of Q .

$$\begin{aligned} P_1 &= K \xrightarrow{1} K \xrightarrow{1} K = I_3 \\ P_2 &= 0 \rightarrow K \xrightarrow{1} K \\ P_3 &= 0 \rightarrow 0 \rightarrow K = E_3 \\ I_1 &= K \rightarrow 0 \rightarrow 0 = E_1 \\ I_2 &= K \xrightarrow{1} K \rightarrow 0 \\ E_2 &= 0 \rightarrow K \rightarrow 0 \end{aligned}$$

Based on the shape of the quiver Q ,

$$c_1 = d_2 - e_2, c_2 = e_1 + d_3 - e_3, c_3 = e_2$$

$CC(P_1)$: Subreps are $0 \rightarrow 0 \rightarrow 0$, $0 \rightarrow 0 \rightarrow K$, $0 \rightarrow K \rightarrow K$, $K \rightarrow K \rightarrow K$.

so $\chi(\text{Gr}_{\vec{e}}(P_1)) = 1$ for $\vec{e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\chi = 0$ otherwise,

$$\Rightarrow CC(P_1) = \frac{x_1^1 x_2^1 x_3^0 + x_1^1 x_2^0 x_3^0 + x_1^0 x_2^0 x_3^1 + x_1^0 x_2^1 x_3^1}{x_1 x_2 x_3}$$

$CC(P_2)$: Subreps are $0 \rightarrow 0 \rightarrow 0$, $0 \rightarrow 0 \rightarrow K$, and $0 \rightarrow K \rightarrow K$. $\chi = 1$ for only corr. dim. vectors

$$\Rightarrow CC(P_2) = \frac{x_1^1 x_2^1 x_3^0 + x_1^1 x_2^0 x_3^0 + x_1^0 x_2^0 x_3^1}{x_2 x_3}$$

$$CC(P_3) = \frac{x_1^0 x_2^1 x_3^0 + x_1^0 x_2^0 x_3^1}{x_3}$$

(13)

as the only subreps (since P_3 is simple)
 are $0 \rightarrow 0 \rightarrow 0$ and P_3 itself

By similar calculations, we obtain

$$CC(I_1) = \frac{x_1^0 x_2^0 x_3^0 + x_1^0 x_2^1 x_3^0}{x_1}$$

$$CC(I_2) = \frac{x_1^0 x_2^1 x_3^1 + x_1^0 x_2^0 x_3^1 + x_1^1 x_2^0 x_3^0}{x_1 x_2}$$

$$CC(E_2) = \frac{x_1^1 x_2^0 x_3^0 + x_1^0 x_2^0 x_3^1}{x_2}$$

We will sketch the proof of
 Caldero-Chapoton's Theorem after
 finishing the proofs of Gabriel's and Kai's
 Theorems.

$$\begin{array}{ccc}
 \left[\begin{smallmatrix} 0 & \\ 0 & 1 \end{smallmatrix} \right] P_3 = E_3 & \left[\begin{smallmatrix} 0 & \\ 0 & 1 \end{smallmatrix} \right] E_2 & E_1 = I_1 \left[\begin{smallmatrix} 1 & \\ 0 & 0 \end{smallmatrix} \right] \\
 \downarrow & \nearrow & \uparrow \\
 \left[\begin{smallmatrix} 1 & \\ 0 & 1 \end{smallmatrix} \right] P_2 & & I_2 \left[\begin{smallmatrix} 1 & \\ 0 & 0 \end{smallmatrix} \right] \\
 \downarrow & \nearrow & \\
 \left[\begin{smallmatrix} 1 & \\ 1 & 0 \end{smallmatrix} \right] P_1 = I_3 & &
 \end{array}$$

Motivating Question: Any guesses why

$$CC\left(\begin{smallmatrix} 0 & \\ \xrightarrow{k^2} & k^2 \\ 0 & 0 \end{smallmatrix}\right) = \left(\frac{x_2 + 1 + x_1}{x_1 x_2}\right)^2 ?$$