

Lecture 18 : Proof of Kac's Theorem,  
 Gregg Musiker 8680 Part I (3-28-11)

Note Title

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① Consider the following two families of quivers:



$\bullet \xrightarrow{m=1}$  Finite type (K alg. closed)

$\bullet \xrightarrow{m=2}$  Kronecker quiver : in 1890 (using different language) Kronecker classified the indecomposables up to isom.

$$K^n \xrightarrow[\substack{I_n \\ J_n(\lambda)}} K^n \quad \lambda \in K \quad \left. \right\} P(K)$$

$$K^n \xrightarrow[\substack{J_n(0) \\ I_n}} K^n \quad " \lambda = \infty "$$

$$K^{n+1} \xrightarrow[\substack{[I_n \\ O] \\ [O \\ I_n]}} K^n \quad \& \quad K^n \xrightarrow[\substack{[I_n \\ O] \\ [O \\ I_n]}} K^{n+1}$$

No other dim vectors have indecomps.  
one param. family, known as tame

$\bullet \xrightarrow{m=3}$  3-Kronecker quiver

Can find a 2-parameter family of pairwise nonisomorphic indecomps this time.

In fact, such a quiver is known as wild, meaning that if we could classify indecomposables for this quiver, we could also classify them for any other quiver.

② For  $n = 2, 3$  or  $4$  of finite type

$n = 5$

Four Subspace problem

Again a one parameter family of pairwise non-isomorphic indecomps. Known as tame.

$n = 6$

Five Subspace problem

Wild!

Another example:

How many indecomps?

One parameter family (infinite)  
so this is also tame.

The finite/tame/wild triality is important in quiver representations.

Tame: # indecomps is infinite, but complete list can be described explicitly. In each dim, there are finitely many one-param. families of pairwise non-isom. indecomps.

Wild: complete list of indecomp is not known

We exhibit this in our discussion of Kac's Theorem below.

(3) For root systems attached to quivers we let  $\Delta$  be a basis of simple roots  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  s.t. any  $\alpha \in \Delta$  can be written as  $\alpha = c_1 \alpha_1 + \dots + c_n \alpha_n$  w/ all  $c_i$ 's  $\geq 0$  or all  $c_i$ 's  $\leq 0$ .

$$\Rightarrow \Delta = \Delta_+ \cup \Delta_-$$

we are in the simply laced case where  $\langle \alpha_i, \alpha_i \rangle = 1$  for each simple root ( $v_i$  loopless)

$$\langle \alpha_i, \alpha_j \rangle := \delta_{ij} - \frac{1}{2} b_{ij} \text{ where } b_{ij} = \sum_{i=j}^{\# \alpha_i}$$

$$\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i, \alpha_j \rangle + \langle \alpha_j, \alpha_i \rangle$$

symmetrized (For quivers,  $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle$ )

Recall that we also can partition  $\Delta$   
 $\Delta = \Delta^{re} \cup \Delta^{Im}$  &  $\Delta_+ = \Delta^{re} \cup \Delta_+^{Im}$   
but we will be a little more methodical.

Let  $\Pi = \underbrace{\{\alpha_i \text{ s.t. vertex } i \text{ has no loop}\}}_{\text{Fundamental roots}}$

Rem:  $\{\text{Fund roots}\} = \{\text{Simple roots}\}$ , if  
 $\mathbb{Q}$  is loopless.

$s_i(\alpha) := \alpha - 2 \langle \alpha, \alpha_i \rangle \alpha$  if  
 $\alpha_i$  is fundamental.

$$\langle \alpha_i, \alpha_j \rangle = \frac{1}{2} (\alpha_i, \alpha_j)$$

which was defined  
last time.

$$\Rightarrow \sum \langle \alpha_i, \alpha_j \rangle = C_{ij} \text{ where } C = \text{Cartan matrix}$$

$$\textcircled{4} \quad \Delta^{\text{re}} := \bigcup_{w \in W} w(\Pi)$$

$$W \in W = \langle s_i : \alpha_i \in \Pi \rangle$$

Def: The Fundamental set is  $M \subset \mathbb{Z}_{\geq 0}^n$

$$M = \left\{ \alpha \in \mathbb{Z}_{\geq 0}^n - \{0\} \mid \langle \alpha, \alpha_i \rangle \leq 0 \text{ for } \alpha_i \in \Pi \right\}$$

and  $\text{supp}(\alpha)$  is connected

Def:  $\Delta^{Im} := \bigcup_{w \in W} w(M \cup -M)$

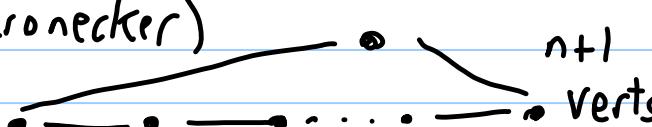
i.e.  $\Delta_+^{Im} = \bigcup_{w \in W} wM, \Delta_-^{Im} = \bigcup_{w \in W} w(-M)$

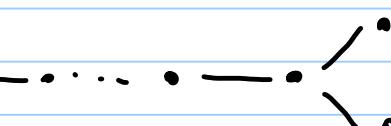
Prop: If there exists  $\delta \in \mathbb{Z}_{\geq 0}^n$  s.t.  
 $\langle \delta, \alpha_i \rangle = 0$  for all  $i$  then  $\Gamma$  is of  
tame type.

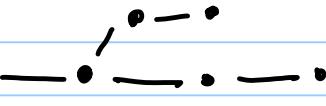
From The HW, you probably have shown (at least in the symmetric case) that  $\Gamma(\Delta)$  is tame if  $\Gamma$  is an extended Dynkin diagram

$\tilde{A}_0 \cdot \circ \cdot$  (a.k.a. affine or Euclidean type)

$\tilde{A}_1 \cdot \circ \cdot$  (Kronecker)

$\tilde{A}_n \quad n \geq 2$   Verts

$\tilde{D}_n \quad n \geq 4$  

$\tilde{E}_6$  

$\tilde{E}_7$  

$\tilde{E}_8$  

⑤ Def: A connected graph of wild type is hyperbolic if every one of its proper connected subgraphs is of finite or tame type.

Prop: Let  $\Gamma$  be connected of finite, tame, or hyperbolic type. Then the root system  $\Delta(\Gamma)$  is the set of nonzero  $\alpha \in \mathbb{Z}^n$  such that  $\langle \alpha, \alpha \rangle \leq 1$ .

Furthermore, if  $\Gamma$  is of finite type,  
 $\Delta(\Gamma) = \{\alpha \in \mathbb{Z}^n : \langle \alpha, \alpha \rangle = 1\}$ .

If  $\Gamma$  is of tame type,

$$\begin{aligned}\Delta^{\text{re}}(\Gamma) &= \{\alpha \in \mathbb{Z}^n : \langle \alpha, \alpha \rangle = 1\}, \\ \Delta^{\text{Im}}(\Gamma) &= \{\alpha \in \mathbb{Z}^n : \langle \alpha, \alpha \rangle \leq 0\} = (\mathbb{Z} - \{0\}) \delta \\ \text{w/ } \delta \text{ satisfying } \langle \delta, \alpha_i \rangle &= 0 \text{ for } \alpha_i \in \Pi.\end{aligned}$$

Pf: First, let  $\alpha = c_1 \alpha_1 + \dots + c_n \alpha_n \in \mathbb{Z}^n - \{0\}$   
s.t.  $\langle \alpha, \alpha \rangle \leq 1$ . We want to show that  $\alpha \in \Delta$ .

Claim:  $\alpha$  must have connected support  
w.r.t.  $\Gamma$  (equivalently, quiver  $Q$ ).

Pf of Claim: Otherwise, we could write  
 $\alpha = \beta + \gamma$  with  $\langle \beta, \gamma \rangle = 0$ .

Also, since any proper connected subgraph of tame type is of finite type,  
we can assume that  $\text{supp } \beta$  and  $\text{supp } \gamma$  are unions of subgraphs of finite type.

Thus by induction on the size of  $\text{supp } \beta$ ,  
we assume  $\langle \beta, \beta \rangle \neq \langle \gamma, \gamma \rangle = 1$   
 $\Rightarrow \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle + \langle \beta, \gamma \rangle + \langle \gamma, \beta \rangle + \langle \gamma, \gamma \rangle$   
 $\geq 1 + 0 + 0 + 1$   
 $\geq 2$ .  $\Rightarrow \alpha \notin$

⑥ Now that  $\alpha$  has conn. support, we wish to show  $\alpha \in \mathbb{Z}_{\geq 0}$  or  $-\alpha \in \mathbb{Z}_{\geq 0}$ .

To see this, let  $\alpha = \beta - \gamma$  with  $\beta, \gamma \in \mathbb{Z}_{\geq 0}$  and  $\text{supp } \beta \cap \text{supp } \gamma = \emptyset$ .

We can assume that  $\text{supp } \beta$  is a union of subgraphs of finite type and  $\text{supp } \gamma$  is either a subgraph of tame type or a union of subgraphs of finite type.

But then

$$\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle - 2\langle \beta, \gamma \rangle + \langle \gamma, \gamma \rangle \text{ with}$$

$$\langle \alpha, \alpha \rangle \leq 1, \text{ and } \langle \beta, \beta \rangle \geq 1, \langle \beta, \gamma \rangle \leq 0$$

$\Rightarrow \langle \gamma, \gamma \rangle \leq 0$ , implying  $\text{supp } \gamma$  is a subgraph of tame type.

But then  $\text{supp } \alpha \nsubseteq \text{supp } \gamma$  connected and the desired inequalities cannot all be satisfied. e.g. led to possibility

$$\langle \beta, \beta \rangle = 1, \langle \gamma, \gamma \rangle = 0, \langle \beta, \gamma \rangle = 0 \text{ which is not possible for these types of graphs.}$$

So w.l.o.g., we can assume  $\langle \alpha, \alpha \rangle \leq 1$ ,  $\text{supp } \alpha$  connected and  $\alpha \in \mathbb{Z}_{>0}^+$ .

We wish to show  $\alpha \in \Delta_+ = \Delta_+^{\text{Re}} \cup \Delta_+^{\text{Im}}$

Claim: with the above hypotheses,  $\alpha$  is real,

i.e. there exists fundamental root  $\alpha_i \in \Pi$  and reflection  $w \in (s_i, -s_n)$  s.t.  $\alpha = w\alpha_i \Rightarrow \langle \alpha, \alpha \rangle = 1$ .

⑦ PF: Recall  $\langle \alpha_i^\vee, \alpha_i \rangle = 1$  for all simple roots (assuming in the simply-laced/symmetric case).

By induction assume  $B$  is real and  $\alpha = s_i B$ ,  $\langle B, B \rangle = 1$ . Then

$$\begin{aligned}\langle s_i B, s_i B \rangle &= \langle B - 2\langle B, \alpha_i \rangle \alpha_i, B - 2\langle B, \alpha_i \rangle \alpha_i \rangle \\ &= \langle B, B \rangle - 2\langle B, \alpha_i \rangle \langle B, \alpha_i \rangle - 2\langle B, \alpha_i \rangle \langle \alpha_i, B \rangle \\ &\quad + 4\langle B, \alpha_i \rangle^2 \langle \alpha_i, \alpha_i \rangle = 1.\end{aligned}$$

To see the reverse implication (i.e.  $\langle \alpha, \alpha \rangle = 1 \Rightarrow \alpha$  is real), we show that  $\alpha$  not real

$\Downarrow$   
 $\alpha$  imaginary  $\Rightarrow \langle \alpha, \alpha \rangle \leq 0$ .

So assume  $W\alpha \cap \Pi = \emptyset$ . Then

$W(\alpha) \subset \bigcup_{i=1}^n$  since only reflection sending pos root to neg root is  
 $s_i \alpha_i = -\alpha_i$ .

We call the height of  $\alpha = c_1 \alpha_1 + \dots + c_n \alpha_n$  the sum  $\sum c_i$ .

pick a  $B \in W\alpha$  of minimal height.

There must exist a simple  $\alpha_i$  s.t.  $\langle B, \alpha_i \rangle > 0$  unless  $B \in \text{Fund Set } M$ .

However then  $s_i B = -2\langle B, \alpha_i \rangle \alpha_i$  is a positive root in  $W\alpha$  of strictly smaller height than  $B$ .  $\Rightarrow \langle =$

⑧ Thus  $\alpha$  not real  $\Rightarrow \alpha = wB$  with  $B \in M \Rightarrow \alpha$  imaginary.

Furthermore  $\langle \alpha, \alpha \rangle = \langle B, B \rangle$  for  $B \in M$   
for same reasoning as above

and  $\langle B, \alpha_i \rangle \leq 0$  for each  $\alpha_i$ ,  
 $\Rightarrow \langle B, B \rangle \leq 0$ .  $\blacksquare$

Finally, description of vector  $\alpha$  agrees with  
 $\langle \cdot, \cdot \rangle$  being pos-semi-definite for tame type.  
This completes the pf of the Prop.

We now wish to show that indecomposables  
only correspond to positive roots.

Theorem [Kac] Let  $K$  be alg. closed  
 $Q$  is a quiver with assoc. root system  $\Delta(\Gamma)$ .

1) There exists an indecomp. rep of  $\underline{\dim} \alpha$   
 $(\Rightarrow \alpha \in \Delta_+(\Gamma))$ . (Note:  $\alpha \neq 0 \in \mathbb{Z}_{\geq 0}^n$ )

2) There exists a unique indecomp. rep  
of  $\underline{\dim} = \alpha \Leftrightarrow \alpha \in \Delta_+^{\text{re}}(\Gamma)$ .

3) If  $\alpha \in \Delta_+^{\text{re}}(\Gamma)$ , then the parameter  
space of  $Q$  reps with  $\underline{\dim} = \alpha$  is  
of dimension  $= 1 - \langle \alpha, \alpha \rangle > 0$ .

Sketch of Proof: By alg. geom,

one can show that For  $\alpha \in \Delta_+$ ,  
the parameter space w/  $\underline{\dim} V = \alpha$   
has dimension  $1 - \langle \alpha, \alpha \rangle$ .

⑨ By above prop,  $\alpha \in \Delta^+ \Rightarrow \langle \alpha, \alpha \rangle \leq 1$   
 and so  $1 - \langle \alpha, \alpha \rangle \geq 0$  with equality  
 $\Leftrightarrow \langle \alpha, \alpha \rangle = 1$ , i.e.  $\alpha$  is real.

Also  $\dim = 1 - \langle \alpha, \alpha \rangle < 0$  so  
 no indecomps of  $\dim$  vector  $\alpha$   
 if  $\langle \alpha, \alpha \rangle > 1 \Rightarrow$  if  $\alpha \in \mathbb{C}$  is outside  
 of the root system.

We will see next time that this  
 parameter space is irreducible so  
 $\dim \neq 0$  does imply a single indecomp.  
 of that dimension vector, up to isom.

Further  $\alpha$  imaginary root implies  
 there are an infinite number of  
 pairwise non-isomorphic indecomposables  
 with  $\dim$  vector  $= \alpha$ .

Back to Kronecker e.g.  $\overset{\circ}{\longrightarrow} \overset{\circ}{\longrightarrow}$

$$\text{Cartan matrix } \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \begin{aligned} c_{ij} &= 2 \langle \alpha_i, \alpha_j \rangle \\ &= (\alpha_i, \alpha_j) \end{aligned}$$

$$\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 1$$

$$\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_1 \rangle = -1$$

$$s_1 \alpha_2 = 2\alpha_1 + \alpha_2, \quad s_2 s_1 \alpha_1 = 2\alpha_1 + 3\alpha_2, \dots$$

$$s_2 \alpha_1 = \alpha_1 + 2\alpha_2, \quad s_1 s_2 \alpha_2 = 3\alpha_1 + 2\alpha_2, \dots$$

$$\begin{aligned} \text{Real pos roots} &= \left\{ n\alpha_1 + (n+1)\alpha_2 : n \geq 0 \right\} \\ &\cup \left\{ (n+1)\alpha_1 + n\alpha_2 : n \geq 0 \right\} \end{aligned}$$

(10) Imag pos roots  $\{n\alpha_1 + n\alpha_2 : n \geq 1\}$

$$\langle c_1\alpha_1 + c_2\alpha_2, \alpha_1 \rangle = c_1 - c_2 \leq 0$$

and

$$\langle c_1\alpha_1 + c_2\alpha_2, \alpha_2 \rangle = c_2 - c_1 \leq 0$$

$$\Rightarrow c_1 = c_2$$

This agrees w/  
w/ dim vectors  
(see pg. 1)  $\begin{bmatrix} n \\ n \end{bmatrix}, \begin{bmatrix} n+1 \\ n \end{bmatrix}, \text{ or } \begin{bmatrix} n \\ n+1 \end{bmatrix}$ .

only  $\begin{bmatrix} n \\ n+1 \end{bmatrix} \text{ & } \begin{bmatrix} n+1 \\ n \end{bmatrix}$ 's lead to  
rigid indecomposables

For each  $n \in \mathbb{N}_{\geq 1}$ ,  $P(k)$  worth of  
pairwise nonisomorphic indecomposables,