

Lecture 19: Imaginary Examples and 3-30-11
Math 8680 Gregg Musiker Kac, Part II

Note Title

3/29/2011

①

OH Tomorrow (Thurs) 2:30-3:20
instead of Friday

We begin with the unproved prop from last time.

Def: A connected graph of wild type is hyperbolic if every one of its proper connected subgraphs is of finite or tame type.

Prop: Let Γ be connected of finite, tame, or hyperbolic type. Then the root system $\Delta(\Gamma)$ coincides with the subset $\{\alpha \in \mathbb{R}^n : \langle \alpha, \alpha \rangle \leq 1\}$.

Further, $\langle \alpha, \alpha \rangle = 1 \iff \alpha \in \Delta^{\text{re}}$
and $\langle \alpha, \alpha \rangle \leq 0 \iff \alpha \in \Delta^{\text{im}}$.

We break down the proof into a few different claims:

Claim 1: If $\text{Supp } \alpha$ does not correspond to a connected subset of Γ , then $\langle \alpha, \alpha \rangle > 1$.

Observation 2: If Γ does not have a loop at i , and α_i is the corresponding simple root, then $\langle \alpha_i, \alpha_i \rangle < 0$.

Claim 2: If $w \in W = (s_1, s_2, \dots, s_n)$, $\langle w\alpha_i, w\alpha_i \rangle = \langle \alpha_i, \alpha_i \rangle$.

Cor: For any real root α , $\langle \alpha, \alpha \rangle = 1$.

② Claim 3: If $\alpha \notin (\mathbb{N}_{\geq 0}^n \cup \mathbb{N}_{\leq 0}^n)$
then $\langle \alpha, \alpha \rangle > 1$.

By Claims 1-3, α satisfying $\langle \alpha, \alpha \rangle \leq 1$
 $\Rightarrow \text{supp } \alpha \text{ connected and}$
 $\alpha \in \mathbb{N}_{>0}^n \text{ or } \mathbb{N}_{\leq 0}^n$.

Claim 4: If α satisfies the following
hypotheses: i) $\alpha \in \mathbb{N}_{>0}^n$,
ii) $\langle \alpha, \alpha \rangle \leq 1$,
iii) $\nexists w \in W \text{ and } \nexists \alpha_i \text{ s.t. } \alpha = w\alpha_i$.
Then $\langle \alpha, \alpha \rangle \leq 0$ and α is a positive
imaginary root.

Rem: A vector $\alpha \in \mathbb{N}_{>0}^n$ is a positive imaginary
root iff $\exists w \in W = \langle s_1, s_2, \dots, s_n \rangle \notin$

$B \in M = \left\{ B \in \mathbb{N}_{>0}^n : \langle B, \alpha_i \rangle \leq 0 \text{ for } \right\}$ s.t.
each α_i AND
 $\text{supp } B \text{ connected}$

$\alpha = wB_0$ [This can be taken as the]
def of imaginary root.

In conclusion, as long as α does not
fail obvious tests [disconnected support,
or having both pos. and neg. coeffs]
then α is either a real or imag.
root.

α is real $\Leftrightarrow \alpha \in W\{\alpha_1, \dots, \alpha_n\}$
 $\Leftrightarrow \langle \alpha, \alpha \rangle = 1$

and imaginary w/ $\langle \alpha, \alpha \rangle \leq 0$ o.w.

[The proof of these claims are on pg 5.
5-8 of Lecture 18 notes.]

③ Let us go through the proof of Claim 4 as it is more surprising:

Pf: We define the height of a positive vector $\alpha = c_1\alpha_1 + \dots + c_n\alpha_n$ as $\sum c_i$.

Let β be a minimal element in the orbit $W\alpha$ w.r.t. height.

[Since (by (iv)) $W\alpha \cap \{\alpha_1, \dots, \alpha_n\} = \emptyset$ and $\alpha \in \mathbb{Z}_{\geq 0}^n$ (by (i)), it follows that

$W\alpha \subset \mathbb{Z}_{\geq 0}^n$, hence why picking a minimal element makes sense]

If B satisfied $\langle B, \alpha_i \rangle > 0$ for some α_i , then

$$S_i B = B - 2\langle B, \alpha_i \rangle \alpha_i = B' \in \mathbb{Z}_{\geq 0}^n$$

if a strictly smaller height.
=>

Thus $\langle B, \alpha_i \rangle \leq 0$ for each α_i .

Also, $\langle B, B \rangle = \langle B, d_1\alpha_1 + \dots + d_n\alpha_n \rangle \leq 0 \Rightarrow$
by claim 1 that $\text{Supp } B$ conn. and
 $B \in M_j$, the fundamental set.

In fact for the tame case, we can see that

$$\left\{ B : \begin{array}{l} \langle B, \alpha_i \rangle \leq 0 \text{ for each } \alpha_i \\ \text{Supp } B \text{ connected} \end{array} \right\}$$

$$= (\mathbb{Z} - \{0\})\delta \text{ for some pos root } \delta$$

(4) and $\langle \delta, \alpha_i \rangle = 0$ for each $\alpha_i \Rightarrow$

$s_i \delta = \delta$ for each reflection.

Example 1: $\bullet \xrightarrow{\quad} \bullet$

$$\langle \alpha_1, \alpha_1 \rangle = 1 = \langle \alpha_2, \alpha_2 \rangle$$

$$\langle \alpha_1, \alpha_2 \rangle = -1 = \langle \alpha_2, \alpha_1 \rangle$$

$$\langle c_1 \alpha_1 + c_2 \alpha_2, \alpha_1 \rangle \leq 0 \text{ &}$$

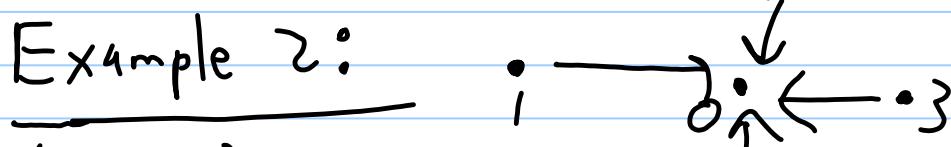
$$\langle c_1 \alpha_1 + c_2 \alpha_2, \alpha_2 \rangle \leq 0 \Rightarrow c_1 = c_2$$

Let $\delta = \alpha_1 + \alpha_2$. For $c \in \mathbb{Z} \setminus \{0\}$,

$$\langle c\delta, \alpha_1 \rangle = 0 = \langle c\delta, \alpha_2 \rangle \Rightarrow$$

$$s_i(c\delta) = c\delta - 2\langle c\delta, \alpha_i \rangle \alpha_i = c\delta.$$

Example 2:



$$\langle \alpha_i, \alpha_0 \rangle$$

$$\langle \alpha_0, \alpha_i \rangle = -1/2 \text{ for } i \in \{1, 2, 3, 4\}$$

$$\langle \alpha_i, \alpha_i \rangle = 1 \text{ for } i \in \{0, 1, 2, 3, 4\}$$

$$\langle \alpha_i, \alpha_j \rangle = 0 \text{ for } i \neq j \in \{1, 2, 3, 4\}$$

Thus, if we want

$$\langle c_0 \alpha_0 + c_1 \alpha_1 + \dots + c_4 \alpha_4, \alpha_i \rangle \leq 0 \text{ for each } i,$$

$$\textcircled{5} \text{ then } \begin{cases} i=0 \\ i=1 \\ i=2 \\ i=3 \\ i=4 \end{cases} c_0 - \frac{1}{2}(c_1 + c_2 + c_3 + c_4) \leq 0,$$

$$i=1) -\frac{1}{2}c_0 + c_1 \leq 0,$$

$$i=2) -\frac{1}{2}c_0 + c_2 \leq 0,$$

$$i=3) -\frac{1}{2}c_0 + c_3 \leq 0,$$

$$i=4) -\frac{1}{2}c_0 + c_4 \leq 0.$$

$$\Rightarrow c_1, c_2, c_3, c_4 \leq \frac{1}{2}c_0 \notin$$

$$c_0 \leq \frac{1}{2}(c_1 + c_2 + c_3 + c_4) \leq \frac{1}{4}c_0 + \frac{1}{4}c_0 + \frac{1}{4}c_0 + \frac{1}{4}c_0 \\ = c_0$$

$$\Rightarrow c_1 = c_2 = c_3 = c_4 = \frac{1}{2}c_0.$$

So we let $\delta = 2\alpha_0 + \alpha_1 + \dots + \alpha_4.$

Fundamental Set $M = \{c_0\delta : c_0 \in \mathbb{Z} - \{0\}\}$

[Supp δ connected] $\iff 1+1$

$$\langle c_0\delta, \alpha_i \rangle = 0 \quad \forall i \in \{1, 2, 3, 4\}$$

$$2-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}$$

$$\text{So again, } \sum_i (c_0\delta) = c_0\delta \\ \text{for } i \in \{0, 1, 2, 3, 4\}.$$

⑥ In fact, we see that

\downarrow
 $\bullet \rightarrow \bullet \leftarrow \bullet$ has some real
 \uparrow roots of the form

$$\begin{array}{c} s_2, s_3, s_4 & s_1, s_3, s_4 & s_1, s_2, s_4 & s_1, s_2, s_3 \\ \textcircled{s}_1 & \textcircled{s}_2 & \textcircled{s}_3 & \textcircled{s}_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_0 \\ \hline s_0 & s_0 & s_2 & s_0 & s_0 & s_4 \end{array}$$

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ + & + & + & + \\ \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 \\ \swarrow & \swarrow & \swarrow & \swarrow \\ \alpha_1 + \alpha_2 & \alpha_2 + \alpha_3 & \alpha_3 + \alpha_4 & \alpha_3 + \alpha_4 \end{array}$$

$$\begin{array}{c} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ + \alpha_0 \end{array} \quad \begin{array}{c} \alpha_2 + \alpha_3 + \alpha_4 \\ + \alpha_0 \end{array}$$

$$\begin{array}{c} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ + \alpha_0 \end{array} \quad \begin{array}{c} \alpha_2 + \alpha_3 + \alpha_4 \\ + \alpha_0 \end{array}$$

$$\begin{array}{c} s_0 \\ s_4 \backslash s_3 \backslash s_2 / s_1 \backslash s_0 \end{array} \quad \begin{array}{c} \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ \dots \end{array}$$

$$\begin{array}{c} \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_0 \\ + \dots \end{array} \quad \begin{array}{c} \alpha_2 + \alpha_3 + \alpha_4 \\ + 2\alpha_0 \end{array}$$

$$\begin{array}{c} s_1, s_2, s_3 \\ \{ \end{array} \quad \begin{array}{c} 3\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ \dots \end{array} \quad \begin{array}{c} s_2, s_3, s_4 \\ \} \end{array}$$

$$\begin{array}{c} \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 \\ + 2\alpha_0 \end{array} \quad \begin{array}{c} 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ + 2\alpha_0 \end{array}$$

⑦ and the infinite set of pos. real roots continues this way.

We can obtain the rigid indecomposables by applying reflection functors to simples of appropriate differently oriented quivers.

E.g. For V w/ $\dim V = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}^0_1_2_3$
 $Q = \begin{array}{c} \overset{2}{\downarrow} \overset{j}{\downarrow} \\ \overset{1}{\rightarrow} \overset{0}{\leftarrow} \overset{-3}{\rightarrow} \\ \overset{1}{\uparrow} \overset{4}{\downarrow} \end{array}$
we apply

$$C_0^- C_1^- C_2^- C_3^- \left(\overset{0}{\downarrow} \overset{\circ}{\downarrow} \overset{0}{\downarrow} \right) =$$

$$C_0^- C_1^- C_2^- \left(\overset{0}{\downarrow} \overset{\circ}{\downarrow} \overset{1}{\rightarrow} \overset{1}{\rightarrow} \right) =$$

$$C_0^- C_1^- \left(\overset{K}{\uparrow} \overset{1}{\rightarrow} \overset{1}{\rightarrow} \overset{0}{\downarrow} \right) =$$

$$C_0^- \left(\overset{1}{\uparrow} \overset{1}{\uparrow} \overset{K}{\uparrow} \overset{1}{\rightarrow} \overset{1}{\rightarrow} \overset{0}{\downarrow} \right) =$$

⑧

K

$$\begin{array}{ccccc}
 & & \downarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & \\
 K & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & \text{Coker } \Psi & \xleftarrow{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} & K \\
 & & \uparrow \circ & &
 \end{array}$$

where $\Psi: V(1) \oplus V(2) \oplus V(3) \rightarrow V(0)$

$$\begin{array}{ccc}
 \text{IIS} & & \text{IIS} \\
 k^3 & & K
 \end{array}$$

$$(a, b, c) \mapsto a+b+c$$

$$\begin{aligned}
 \Rightarrow \text{coker } \Psi &\cong K^2 \cong K^3 / \text{Im } \Psi \\
 &\cong \{(a, b, c) : a+b+c \sim 0\} \\
 &\cong \{(a, b) : a, b \in K\}
 \end{aligned}$$

and can think of maps

$$\begin{array}{ccc}
 V(1) \rightarrow \text{coker } \Psi, & V(2) \rightarrow \text{coker } \Psi \\
 a \mapsto (a, 0) & b \mapsto (0, b)
 \end{array}$$

$$\begin{array}{ccc}
 V(3) \rightarrow \text{coker } \Psi \\
 c \mapsto (-a, -b) \cdot
 \end{array}$$

⑨ For imaginary roots, e.g.

$$\delta = 2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$$

Possible indecomposables, up to isomorphism look like

(K alg closed)

$$\begin{array}{ccccc}
 & & K & & \\
 & & \downarrow \begin{bmatrix} 0 \\ i \end{bmatrix} & & \\
 K & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & K^2 & \xleftarrow{\begin{bmatrix} -1 \\ -i \end{bmatrix}} & K \\
 & & \uparrow \begin{bmatrix} 1 \\ h \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ i \end{bmatrix} \text{ for } h \in K. & & \\
 & & K & &
 \end{array}$$

Pf: Let V be the rep where
 $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \end{bmatrix}$ denotes
the injections $K \rightarrow K^2$.

Note that if any of these
maps were $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, V would be
decomposable

$$\begin{array}{ccccc}
 & & K & & \\
 & & \downarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} & & \\
 K & \rightarrow & 0 & \leftarrow 0 & \oplus \\
 & & \uparrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} & & \\
 & & K^2 & \xleftarrow{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} & K \\
 & & \uparrow \begin{bmatrix} 1 \\ h \end{bmatrix} & & \\
 & & K & \xrightarrow{\begin{bmatrix} 1 \\ h \end{bmatrix}} & K
 \end{array}$$

Thus can assume all 4 vectors $\neq 0$.

⑩ Assume w.l.o.g. $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

If d, f, h were all zero, then

$$V \cong K \xrightarrow{\downarrow c} K \xleftarrow{e} K \oplus 0 \xrightarrow{\downarrow} 0 \xleftarrow{f} 0$$

$\uparrow g$

\Rightarrow w.l.o.g. $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (up to isom)

$$V \cong \left\{ \begin{bmatrix} 0 \\ d \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \right\} \rightarrow K^2 \leftarrow \left\{ \begin{bmatrix} e \\ f \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} g \\ h \end{bmatrix} \right\}$$

.

If $f = h = 0$ (equiv. $e = g = 0$)

then

$$V \cong \begin{array}{c} 0 \\ \downarrow \\ K \xrightarrow{\downarrow} K \xleftarrow{e} K \oplus 0 \xrightarrow{\downarrow} 0 \xleftarrow{f} 0 \\ \uparrow g \\ K \end{array}$$

\Rightarrow up to isom, $\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

⑪ Then $\begin{bmatrix} g \\ h \end{bmatrix}$ can be chosen freely, and we will have an indecomp

So up to isom $\begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} 1 \\ h \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 3:

$$\langle \alpha_1, \alpha_1 \rangle = \delta_{11} - (\# \text{ loops at } 1) = 0$$

$$\begin{array}{cc} // & // \\ 1 & -1 \end{array}$$

Cartan Matrix $[0]$.

$$\begin{aligned} s_1 \alpha_1 &= \alpha_1 - 2\langle \alpha_1, \alpha_1 \rangle \alpha_1 \\ &= \alpha_1 \left[\begin{array}{l} \text{Notice that} \\ s_1 \alpha_1 \neq -\alpha_1 \end{array} \right] \\ &\quad \text{for this quiver} \end{aligned}$$

Also α_1 not a fundamental root since loop at i.

\Rightarrow No real roots and imaginary roots are $\{c\alpha_1 : c \in \mathbb{K} - \{0\}\}$ i.e. $\delta = \alpha_1$.

Kac's Theorem works here as no rigid indecomposables

For any $n \in \mathbb{Z}_{>0}$, $Q = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$

(12) $K^n \circ \hookrightarrow J_n(\lambda) = \begin{bmatrix} \lambda & & & \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & & & \ddots \end{bmatrix}$

pairwise nonisomorphic for different λ .
 (assuming K alg closed)

Review of HW Prob #4

$$1 \rightarrow 2 \leftarrow 3$$

$$\begin{array}{c|c} P_1 \ K \rightarrow k \leftarrow 0 & I_1 \ K \rightarrow 0 \leftarrow 0 \\ P_2 \ 0 \rightarrow k \leftarrow 0 & I_2 \ K \rightarrow k \leftarrow K \\ P_3 \ 0 \rightarrow k \leftarrow K & I_3 \ 0 \rightarrow 0 \leftarrow K \end{array}$$

$$C^- = C_2^- C_3^- C_1^- = C_2^- C_1^- C_3^-$$

$$C^+ = C_3^+ C_1^+ C_2^+ = C_1^+ C_3^+ C_2^+$$

$$C^+ P_i = 0, \quad C^- I_i = 0$$

$$\begin{array}{l|l} C^- P_1 = I_3 & C^+ I_1 = P_3 \\ C^- P_2 = I_2 & C^+ I_2 = P_2 \\ C^- P_3 = I_1 & C^+ I_3 = P_1 \end{array}$$

Same as action of Coxeter elements on the appropriate Jim vector is = pos roots.

(13) Bipartite Belt

$$x_1 \longrightarrow x_2 \longleftarrow x_3$$

$$\downarrow m_2$$

$$\left\{ x_1, \frac{x_1 x_3 + 1}{x_2}, x_3 \right\}$$

$$\downarrow m_1$$

$$\downarrow m_3 \begin{matrix} \text{Mutations} \\ \text{commute} \end{matrix}$$

$$\left\{ \frac{x_1 x_3 + x_2 + 1}{x_1 x_2}, \frac{x_1 x_3 + 1}{x_2}, \frac{x_1 x_3 + x_2 + 1}{x_2 x_3} \right\}$$

P_1

P_2

P_3

$$\downarrow m_2, \text{ then } m_1 \circ m_3$$

$$\left\{ \frac{x_2 + 1}{x_3}, \frac{x_2^2 + x_1 x_3 + 2 x_2 + 1}{x_1 x_2 x_3}, \frac{x_2 + 1}{x_1} \right\}$$

I_3

I_2

I_1

$$\downarrow m_2, \text{ then } m_1 \circ m_3$$

$$\left\{ x_3, x_2, x_1 \right\}$$

Mimics reflection functors.

Also mutating along bipartite belt turns sinks \leftrightarrow sources.