

① Today we begin with the general definition of a cluster algebra.

We start w/ the definition of a labeled seed.

Let \mathbb{P} denote a semifield with addition \oplus and multiplication denoted as usual.

A semifield is like a field, except there are no additive inverses.

\mathbb{P} can also be thought of as an abelian group without torsion.

Cluster Algebra $A \subset \text{Frac}(\mathbb{Z}\langle \mathbb{P}[x_1, \dots, x_n] \rangle)$.

Def: A labeled seed $(\underline{X}, \underline{Y}, B)$ cluster algebra A of rank n is a triple

$\underline{X} = \{x_1, x_2, \dots, x_n\}$ initial cluster [alg ind]

$\underline{Y} = \{y_1, y_2, \dots, y_n\}$ initial $\in \mathbb{P}$ coefficients

$B = n \times n$ skew-symmetrizable matrix

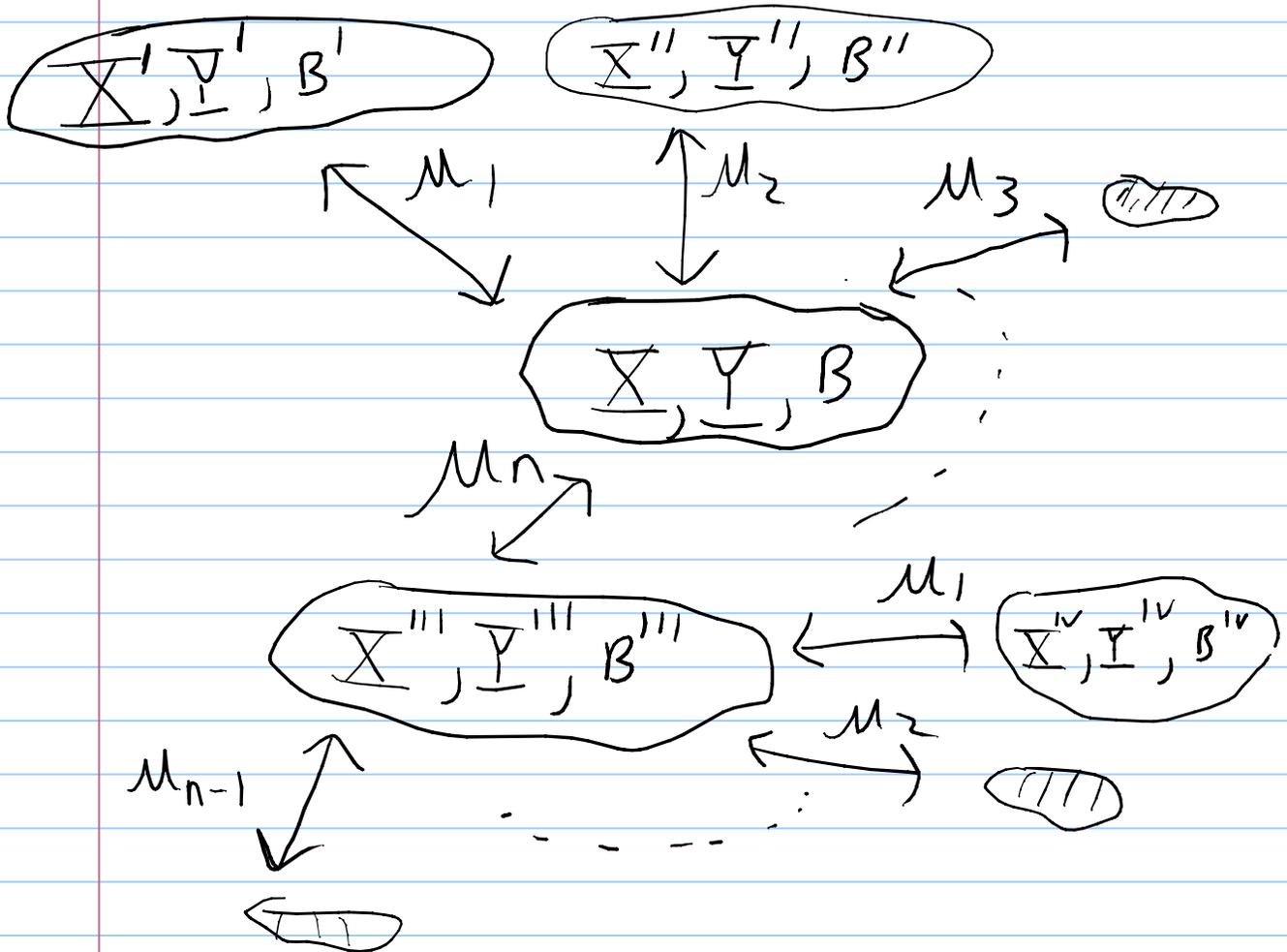
(there exist pos integers d_1, \dots, d_n s.t. $d_i b_{ij} = -d_j b_{ji}$ for $1 \leq i, j \leq n$.)

② Simplest Example (Coefficient-Free and Rank 2):

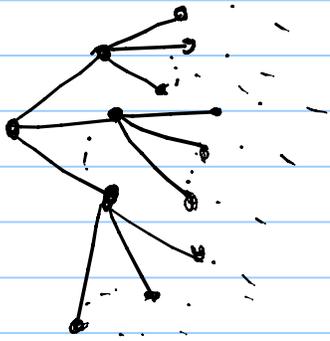
$$B = \begin{bmatrix} 0 & b \\ -c & a \end{bmatrix}, P = \{1\}$$

$$\underline{X} = \{x_1, x_2\}, \underline{Y} = \{y_1, y_2\} = \{1, 1\}$$

A labeled seed gives initial data for a cluster algebra and encodes a mutation rule that can be used to generate other labeled seeds:

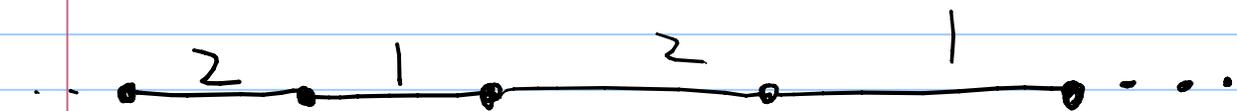


③ In general, this procedure can give rise to an n -ary tree



but in practice, often will contain cycles.

Mutation in the coeff-free rank 2 case modeled by labeled seeds



$$\begin{array}{ccccc} \{x_1, x_0\} & \{x_1, x_2\} & \{x_3, x_2\} & \{x_3, x_4\} & \{x_5, x_4\} \\ -B & B & -B & B & -B \end{array}$$

$$(*) \quad X_n X_{n-2} = \begin{cases} X_{n-1}^b + 1 & \text{if } n \text{ even} \\ 1 + X_{n-1}^c & \text{if } n \text{ odd} \end{cases}$$

Recall that $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$.

As indicated by above tree, we can mutate in the 1st or 2nd direction from each seed.

If we mutate in i th direction, we read down i th col to obtain exchange rule (*), and the new exchange matrix $M_i(B)$ is obtained by multiplying i th row & i th col by (-1) . In this spec. case $\Rightarrow M_i(B) = -B$.

④ Here we give general def:

Def: (Seed Mutation) For any $k \in \{1, 2, \dots, n\}$ ($n = \text{rank } A$), we define $\mu_k(\underline{X}, \underline{Y}, B) = \underline{X}', \underline{Y}', B'$ as

$$\underline{X}' = \{x_1, x_2, \dots, x_{k-1}, x_k', x_{k+1}, \dots, x_n\}$$

$$x_k' := \frac{y_k \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}}{x_k (y_k \oplus 1)}$$

$$\underline{Y}' = \{y_1', y_2', \dots, y_n'\} \text{ where}$$

$$y_j' := \begin{cases} y_j \left(\frac{y_k^{\max(b_{kj}, 0)}}{(y_k \oplus 1)^{b_{kj}}} \right) & \text{if } j \neq k, \\ y_k^{-1} & \text{if } j = k, \end{cases}$$

and $B' = [b_{ij}']$ with

$$b_{ij}' = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } j, \\ b_{ij} + |b_{ik}| b_{kj} & \text{if } k \neq i, j, \\ & \text{and } b_{ik} \cdot b_{kj} > 0, \\ b_{ij} & \text{otherwise.} \end{cases}$$

Note how this rule agrees w/ coeff-free rank 2 case.

⑤ Example: Let $P = \text{Trop}(u_1, u_2, \dots, u_5)$ where

$$u_1^{d_1} \cdots u_5^{d_5} \oplus u_1^{e_1} \cdots u_5^{e_5} := u_1^{\min(d_1, e_1)} \cdots u_5^{\min(d_5, e_5)}$$

$$A = A(\{x_1, x_2\}, \{y_1, y_2\}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) \text{ with}$$

$$y_1 = u_2^{-1} u_4 u_5, \quad y_2 = u_1 u_3^{-1} u_4^{-1}.$$

Let S denote this initial seed.

$$S = (\{x_1, x_2\}, \left\{ \frac{u_4 u_5}{u_2}, \frac{u_1}{u_3 u_4} \right\}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$$

$$\mu_1(S) = (\{x_3, x_2\}, \left\{ \frac{u_2}{u_4 u_5}, \frac{u_1 u_5}{u_3} \right\}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})$$

$$\mu_2 \mu_1(S) = (\{x_3, x_4\}, \left\{ \frac{u_1 u_2}{u_4}, \frac{u_3}{u_1 u_5} \right\}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$$

$$\mu_1 \mu_2 \mu_1(S) = (\{x_5, x_4\}, \left\{ \frac{u_4}{u_1 u_2}, \frac{u_2 u_3}{u_5} \right\}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$$

$$\mu_2 \mu_1 \mu_2 \mu_1(S) = (\{x_2, x_4\}, \left\{ \frac{u_3 u_4}{u_1}, \frac{u_5}{u_2 u_3} \right\}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix})$$

$$\mu_1 \mu_2 \mu_1 \mu_2 \mu_1(S) = (\{x_2, x_1\}, \left\{ \frac{u_1}{u_3 u_4} = x_2, \frac{u_4 u_5}{u_2} = y_1 \right\}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$$

where
$$x_3 = \frac{y_1 + x_2}{x_1(y_1 \oplus 1)} = \frac{\frac{u_4 u_5}{u_2} + x_2}{x_1 \left(\frac{1}{u_2} \right)}$$

$$= \frac{u_4 u_5 + u_2 x_2}{x_1}.$$

⑥ we get the new coefficients as

$$\begin{aligned} y_1' &= y_1^{-1} \\ &= \frac{u_2}{u_4 u_5} \end{aligned} \quad y_2' = y_2 \left(\frac{y_1}{y_1 \oplus 1} \right) = \frac{u_1}{u_3 u_4} \left[\frac{u_4 u_5 / u_2}{1 / u_2} \right] = \frac{u_1 u_5}{u_3}$$

Exercise

$$x_4 = \frac{y_2' + x_3}{x_2 (y_2' \oplus 1)} = \dots$$

$$= \frac{u_2 u_3 x_2 + u_3 u_4 u_5 + u_1 u_5 x_1}{x_1 x_2}$$

Exercise: Verify that y_1'', y_2'' agree with the above, and show

$$x_5 = (u_3 u_4 + u_1 x_1) / x_2, \text{ and the rest of the above table.}$$

We now discuss a second way to think about cluster algebras like the one above, i.e. when

$$\mathbb{P} = \text{Trop}(u_1, u_2, \dots, u_m).$$

Such cluster algebras are known as being of geometric type.

⑦ Let $\tilde{B} = (m+n) \times n$ matrix,
 s.t. $B =$ top $n \times n$ submatrix
 and (skew-symmetrizable),

$$\tilde{X} = \{x_{11}, \dots, x_{n1}, x_{n+11}, \dots, x_{n+m1}\}$$

Then the pair (\tilde{X}, \tilde{B}) is a
labeled seed for a cluster algebra
of geometric type.

x_{11}, \dots, x_{n1} known as exchangeable
 variables,

$x_{n+11}, \dots, x_{n+m1}$ " " frozen variables
 (or a.k.a coefficients)

Let $\tilde{X}' = \mu_k(\tilde{X})$, $\tilde{B}' = \mu_k(\tilde{B})$.

Then $\tilde{X}' = \tilde{X} - \{x_k\} \vee \{x_k'\}$

where we read down columns of \tilde{B}
 to get exchange rule

$$x_k' = \frac{\prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}}{x_k}$$

$(1 \leq i \leq m+n)$

x_k

⑧ and \tilde{B}' defined just as in the $n \times n$ case.

Exercise: Prove that the

mutation rules for \tilde{X}, \tilde{B} agree with that of a general cluster seed $\underline{X}, \underline{Y}, B$ where

$$\underline{X} = \{x_1, \dots, x_n\}$$

$$\underline{Y} = \{y_1, \dots, y_m\} \subset \mathbb{P} = \text{Trop}(x_{n+1}, \dots, x_{n+m})$$

$$\text{with } y_j := \prod_{i=1}^m x_{n+i}^{b_{ij}}$$

and $B = \text{top } n \times n \text{ matrix of } \tilde{B}$.

Example: $\tilde{X} = \{x_1, x_2, u_1, u_2, u_3, u_4, u_5\}$

$$\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\cdot \mu_1(\tilde{X}, \tilde{B}) = \left(\{x_3, x_2, u_1, \dots, u_5\}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \right)$$

9

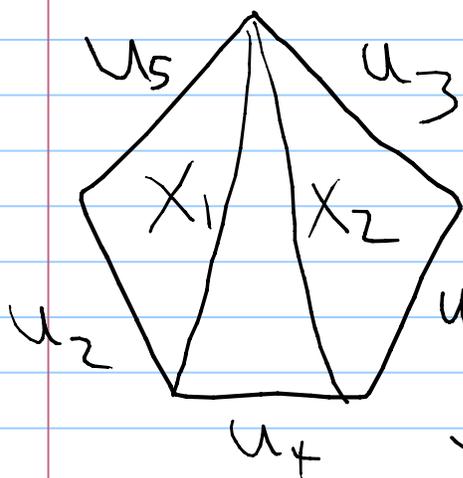
Thus we would have the exchange

$$X_4 = \frac{u_1 u_5 + u_3 X_3}{X_2} \quad \text{which}$$

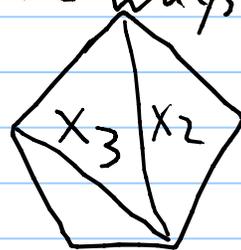
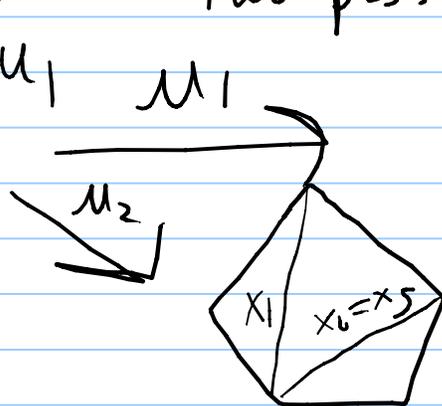
$$\text{agrees w/ } X_4 = \frac{y_2' + X_3}{X_2 (y_2' \oplus 1)}$$

w/ $y_2' = \frac{u_1 u_5}{u_3}$) and the other mutations also agree.

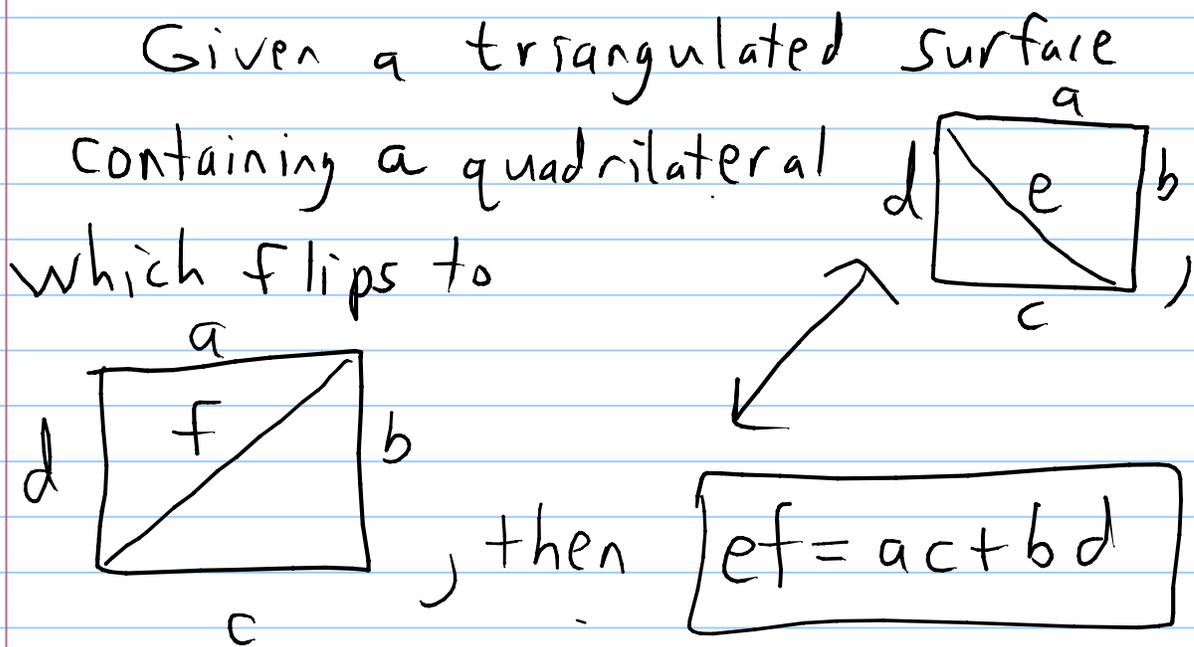
A geometric model for this example: Consider a triangulated pentagon with the following edge labels:



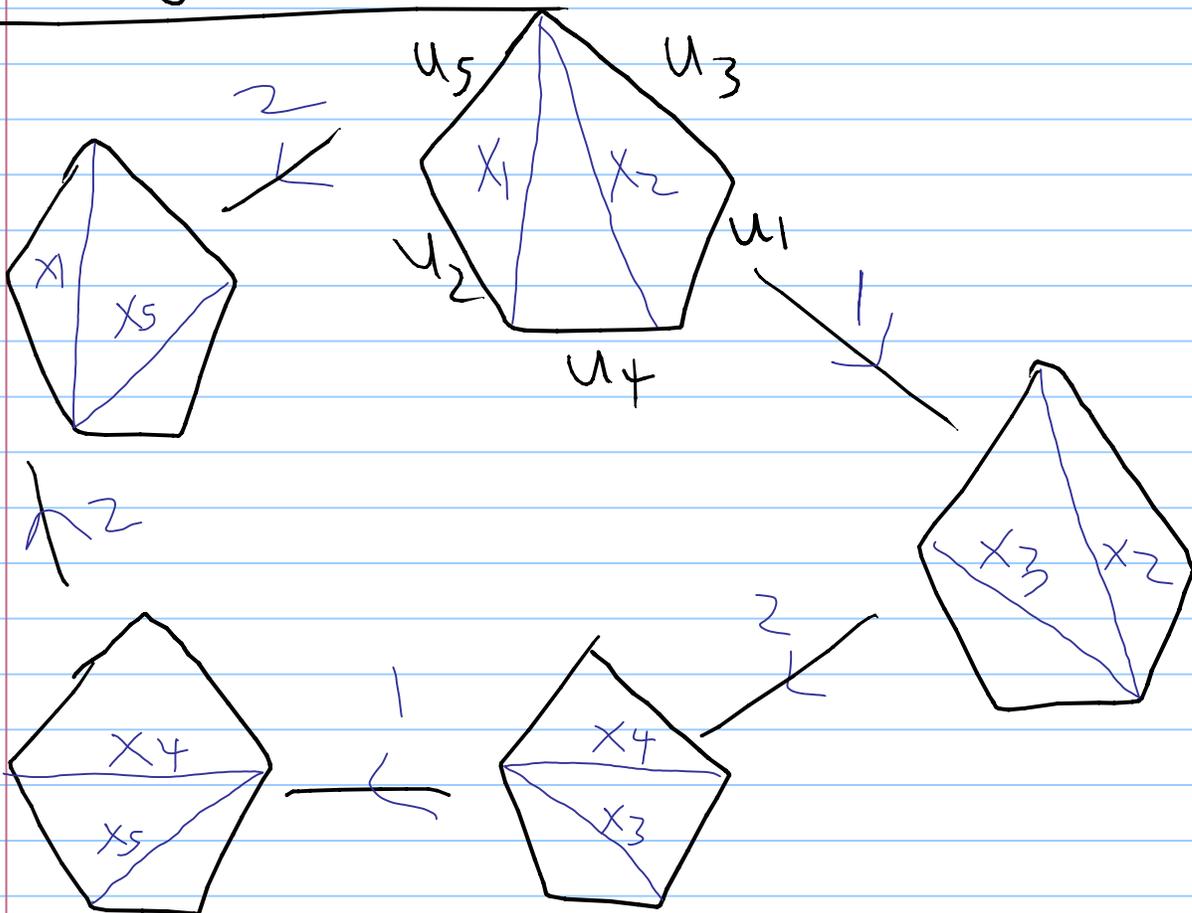
We can Flip this triangulation to a different one in two possible ways



⑩ We make this geometric notion an algebraic one by the Ptolemy Exchange relation:



Pentagon Example



⑪ Using the Ptolemy Exchange Relation,

$$X_3 = \frac{u_2 X_2 + u_4 u_5}{X_1})$$

$$X_4 = \frac{u_3 X_3 + u_5 u_1}{X_2} = \frac{u_3 u_2 X_2 + u_3 u_4 u_5 + u_5 u_1 X_1}{X_1 X_2})$$

$$X_5 = \frac{u_4 X_4 + u_1 u_2}{X_3} =$$

$$\frac{u_4 (u_3 u_2 X_2 + u_3 u_4 u_5 + u_5 u_1 X_1) + u_1 u_2 X_1 X_2}{X_1 X_2 \left(\frac{u_2 X_2 + u_4 u_5}{X_1} \right)}$$

= (after surprising cancellation of polys)

$$\frac{u_3 u_4 + u_1 X_1}{X_2}$$

Notice we get the consistent result

$$X_2 X_5 = u_3 u_4 + u_1 X_1, \text{ agreeing w/ the last quadrilateral.}$$