

Lecture 20: Last Part of Gabriel's Theorem Gregg Musiker 8680 (4-4-II)

Note Title

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①

Alternative Descriptions of $\langle \cdot j \cdot \rangle_Q$:

Crash Course on Alg. Geom. & Quiver Reps
(Sec. 4 of Barot or Lecs. 4 & 5 of Derksen, etc.)

Let Q be a quiver with no oriented cycles or loops, $Q_0 = \{j_1, j_2, \dots, j_n\}$.

Fix a dimension vector $\vec{d} \in \mathbb{N}^n$,

A Q -rep V with $\underline{\dim} V = \vec{d}$ can be completely determined by the $d_j \times d_i$ matrices ($\overrightarrow{a: i \rightarrow j}$) for each arrow $a: i \rightarrow j$.

Let $\text{rep}(Q, \vec{d}) = \prod_{i \rightarrow j \text{ in } Q} K^{d_j \times d_i}$

parametrizing the entries of each of these matrices.

$\text{rep}(Q, \vec{d})$ is the set of all Q -reps V with $\underline{\dim} V = \vec{d}$.

Two reps $V, W \in \text{rep}(Q, \vec{d})$ are isomorphic to one another $\Leftrightarrow \exists$ basis changes on each K^{d_i} so that V and W agree, i.e. \exists a family of invertible maps

$$\{f_i: K^{d_i} \rightarrow K^{d_i}\}_{i \in Q_0} \text{ s.t.}$$

$$V(a)f_i = f_j W(a) \text{ for each } a: i \rightarrow j \in Q_1.$$

Let $GL(Q, \vec{d}) = \prod_{i \in Q_0} GL(d_i)$ and define a group action for each

$$\textcircled{2} \quad g \in GL(Q, \vec{d}), \quad g \circ V = \prod_{a: i \rightarrow j} (g; V(a) g_i^{-1})_a$$

Then $V \cong W$ if and only if
as elts of $\text{rep}(Q, \vec{d})$, they lie in
the same orbit of $GL(Q, \vec{d})$ acting
on $\text{rep}(Q, \vec{d})$.

Let us now do some rudimentary dimension counting:

$$\dim_K(\text{rep}(Q, \vec{d})) = \sum_{a: i \rightarrow j} d_j \cdot d_i$$

$$\dim_K(GL(Q, \vec{d})) = \sum_{i \in Q_0} d_i^2$$

$$\Rightarrow \dim GL(Q, \vec{d}) - \dim \text{rep}(Q, \vec{d})$$

$$= \sum_{i \in Q_0} d_i^2 - \sum_{\substack{a: i \rightarrow j \\ \text{in } Q_1}} d_j d_i .$$

Does this quantity look familiar?

Recall: For loopless quiver Q and

$$\alpha = c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_n \alpha_n$$

$$\beta = d_1 \alpha_1 + d_2 \alpha_2 + \dots + d_n \alpha_n$$

$$\langle \alpha, \beta \rangle := \sum_{1 \leq i, j \leq n} c_i d_j \langle \alpha_i, \alpha_j \rangle \text{ where}$$

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 1 & \text{if } i = j \\ -(\# i \rightarrow j \text{ or } j \rightarrow i)/2 & \text{if } i \neq j \end{cases}$$

$$\textcircled{3} \quad \text{Thus, } \langle \alpha, \alpha \rangle =$$

$$\sum_{i=1}^n c_i^2 \underbrace{\langle \alpha_i, \alpha_i \rangle}_{\substack{|| \\ 1}} + 2 \sum_{1 \leq i < j \leq n} c_i c_j \underbrace{\langle \alpha_i, \alpha_j \rangle}_{\substack{|| \\ -\#(i \rightarrow j \text{ or } j \rightarrow i)}}$$

$$= \dim GL(Q, \alpha) - \dim \text{rep}(Q, \alpha).$$

We now finish Gabriel's Thm:

If $\langle \cdot, \cdot \rangle$ is not pos definite,
 $\exists \vec{d} \neq 0$ s.t. $\langle \vec{d}, \vec{d} \rangle \leq 0$

(Recall that we may assume that
such a \vec{d} has only nonnegative coeffs)
Since o.w. $\vec{d} = \vec{d}^+ - \vec{d}^-$ with $d_i^+ d_i^- = 0$ for
each i , and $\langle \vec{d}, \vec{d} \rangle \leq 0 \Rightarrow \langle \vec{d}^+, \vec{d}^+ \rangle \leq 0$
or $\langle \vec{d}^+, \vec{d}^- \rangle \leq 0$.

$$\langle \vec{d}^+ - \vec{d}^-, \vec{d}^+ - \vec{d}^- \rangle = \langle \vec{d}^+, \vec{d}^+ \rangle + \langle \vec{d}^-, \vec{d}^- \rangle + \sum_{i,j} q_{ij} \vec{d}_i^+ \vec{d}_j^-$$

where $q_{ij} = \#(i \rightarrow j \text{ or } j \rightarrow i)$.

$$\Rightarrow \dim GL(Q, \vec{d}) \leq \dim \text{rep}(Q, \vec{d}).$$

|| By orbit-stabilizer Theorem

$\dim \mathcal{O}(V) + \dim GL(Q, \vec{d})_V$ (for $V \in \text{rep}(Q, \vec{d})$),
where $\mathcal{O}(V) = \text{orbit of } V$,
thought of a point in $\text{rep}(Q, \vec{d})$,
under the action of $GL(Q, \vec{d})$;

and $GL(Q, \vec{d})_V$ denotes the stabilizer
subgroup of V $\{g \in GL(Q, \vec{d}): g \cdot V = V\}$.

(4) For each V , $\dim GL(Q, \vec{d})_V \geq 1$ since

$\lambda \in k$ induces an isomorphism $\in GL(Q, \vec{d})_V$
 $\lambda \cdot V = V$ by basis changes

$b_i \mapsto \lambda \cdot b_i$ for each $b_i \in \prod_k k^{d_i \times d_i}$.

$\Rightarrow \dim O(V) < \dim \text{rep}(Q, \vec{d})$
strict inequality.

and the only way $\{O(V)\}'s\}$ can cover $\text{rep}(Q, \vec{d})$ is if there are infinite number of orbits, i.e. pairwise nonisom. reps of that dim vector.

Conclusion: Q of finite rep type $\Rightarrow \langle \cdot, \cdot \rangle_Q$ pos definite.

Now assume that $\langle \cdot, \cdot \rangle_Q$ is pos definite, which we know from HW 2, is true $\Leftrightarrow Q$ is of type A_n, D_n , or E_6, E_7, E_8 .

We showed last week that

$\langle \alpha, \alpha \rangle = 1$ iff α is a real root
and $\langle \alpha, \alpha \rangle > 1$ if α is not in the root system Δ_Q .

Furthermore, we saw that there are a finite number of (pos real) roots
 $\Leftrightarrow Q$ is of type ADE.

⑤ Also, assuming that Q is of type ADE,
we have a bijection (via reflection
functors) between positive roots and
indecomposable Q -reps.

Conclusion : $\langle \cdot, \cdot \rangle_Q$ pos definite
 $\Rightarrow Q$ is of finite rep type.

Completes the proof of Gabriel's Thm.

I want to now introduce a 3rd
way to think about $\langle \cdot, \cdot \rangle$ that
will be useful later. [Actually a
non-sym. version]

Extensions :

Df: An extension of M by N
is a short exact sequence

$$0 \rightarrow M \xrightarrow{f} L \rightarrow N \rightarrow 0$$

Say that M "extends" to L & $L/f(M) \cong N$.

Two extensions are equivalent if
there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & L & \rightarrow & N \rightarrow 0 \\ & & \downarrow & \phi \downarrow & \downarrow & & \\ 0 & \rightarrow & M & \rightarrow & L' & \rightarrow & N \rightarrow 0 \end{array} \quad \begin{array}{l} \text{Forces } \phi \text{ to} \\ \text{be an } \cong, \text{ but} \\ \text{stronger cond.} \\ \text{than } L \cong L' \end{array}$$

Def: Let $\text{Ext}'(N, M)$ denote the
equivalence classes of extensions of
 M by N .

⑥ We can actually define $\text{Ext}^i(N, M)$ for all $i \geq 0$.

This is done using the Ringel Resolution:

We start with the notion of a projective resolution:

An exact sequence

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow V \rightarrow 0$$

is called a proj. res. for V if all the P_i 's are projective modules.

For quiver reps (ie. kQ -modules) we can use the Ringel res.

$$0 \rightarrow \bigoplus_{a \in Q_1} P_{ta} \otimes V(sa) \xrightarrow{\psi} \bigoplus_{i \in Q_0} P_i \otimes V(i) \rightarrow V \rightarrow 0$$

where the maps $(\begin{smallmatrix} a & \leftarrow \\ ta & sa \end{smallmatrix})$

$$P_i \otimes V(i) \xrightarrow{\psi_i} V \text{ defined by}$$

$$e_j \otimes v \mapsto v.$$

Recall that knowing where $e_i \in P_i(i)$ is sent determines the rest of the map on P_i .

⑦ The map $\bigoplus_{a \in Q_1} P_{ta} \otimes V(sa) \xrightarrow{\psi} \bigoplus_{i \in Q_0} P_i \otimes V(i)$

is given by ψ

$$e_{ta} \otimes v \mapsto e_{ta} \otimes av - a \otimes v$$

with $e_{ta} \otimes v \in P_{ta}(ta) \otimes V(sa)$

$$e_{ta} \otimes av \in P_{ta}(ta) \otimes V(ta)$$

$$a \otimes v \in P_{sa}(ta) \otimes V(sa)$$

(Recall $P_i := KQe_i$)
 $V(j) := e_j V$
and $P_i(j) = e_j V e_i$)

i.e. $P_{sa}(ta) = e_{ta} KQe_{sa} \ni a$

Claim 1: $\bigoplus_{i \in Q_0} \psi_i$ is surjective

Pf: For $v = v_1 + \dots + v_n \in \bigoplus_{i \in Q_0} e_i V = V$
 $\sum e_i \otimes v_i \in \bigoplus_{i \in Q_0} P_i \otimes V(i) \mapsto v.$

Claim 2: The composition of
 $(\bigoplus \psi_i) \circ \psi$ is zero.

Pf: For $e_{ta} \otimes v \in P_{ta}(ta) \otimes V(sa)$,

$$\psi(e_{ta} \otimes v) = e_{ta} \otimes av - a \otimes v \Rightarrow$$

$$(\bigoplus \psi_i) \circ \psi(e_{ta} \otimes v) = av - av = 0$$

since each summand $\mapsto 0$ map is zero.

⑧ Claim 3: Ψ is injective.

Pf: Suppose $\sum_{a \in Q_1} p_a \otimes v_a \in \ker \Psi$.

$$\Psi\left(\sum_{a \in Q_1} p_a \otimes v_a\right) = \sum_{a \in Q_1} p_a \otimes a v_a - a p_a \otimes v_a$$

If Q has no oriented cycles,
label the vertices of Q_0 by $\{1, 2, \dots, n\}$
s.t. $i \rightarrow j \Rightarrow$ no arrow from $i \rightarrow j$.
(n labels the sink, 1 the source)

Suppose $v_a \neq 0$ for some a , and choose
an arrow a such that s_a is the largest
element of $Q_0 \leftrightarrow \{1, 2, \dots, n\}$ with $v_a \neq 0$.

Then since $\sum_{a \in Q_1} p_a \otimes a v_a - a p_a \otimes v_a = 0$

and $p_a \otimes a v_a \in P_{t_a} \otimes V(t_a)$
can only cancel with another

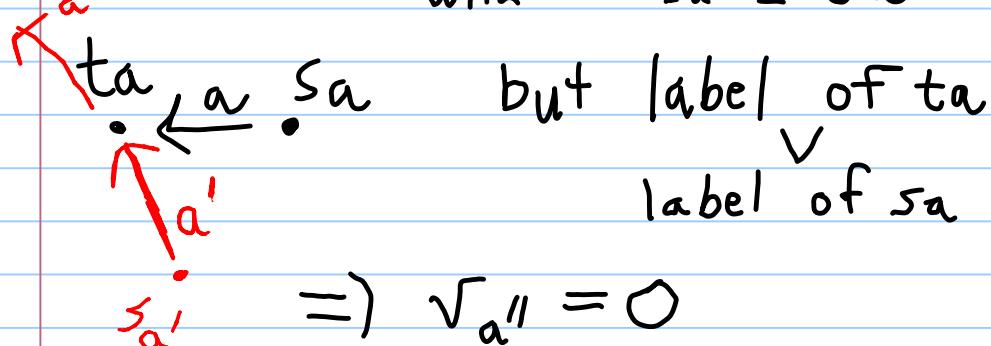
$$P_{a'} \otimes a' v_{a'} \in P_{t_{a'}} \otimes V(t_{a'})$$

with $t_{a'} = t_a$

or an element

$$P_{a''} \otimes v_{a''} \in P_{s_{a''}} \otimes V(s_{a''})$$

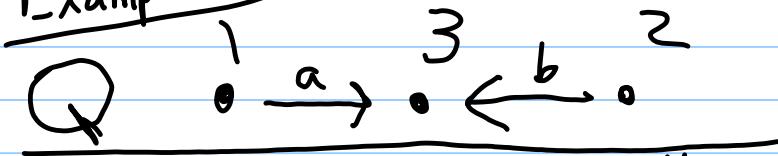
with $s_{a''} = t_a$



and label $s_a >$ label $s_{a'}$ by assumption

(9) so v_a' could be nonzero but soon will exhaust all possible sources
 $\Rightarrow \subset$ [Thus all v_a' 's = 0]

Example



$$P_a \otimes v_a + P_b \otimes v_b \xrightarrow{\psi} 0 =$$

$v(1)$ $v(2)$

$$0 = P_a \otimes a v_a - a P_a \otimes v_a + P_b \otimes b v_b - b P_b \otimes v_b$$

$$\begin{aligned} &= P_a \otimes a v_a + P_b \otimes b v_b \in P_3 \otimes V(3) \\ &\quad - a P_a \otimes v_a \in P_1 \otimes V(1) \\ &\quad - b P_b \otimes v_b \in P_2 \otimes V(2) . \end{aligned}$$

Last two summands must be zero
 $\Rightarrow v_a = 0 \wedge v_b = 0$

Even though from first summand, looks like we have more flexibility,

Claim 4: $\text{Ker}(\bigoplus_i p_i) \subseteq \text{Im } \psi$.

Note: we already showed

$$\text{Ker}(\bigoplus_i p_i) \supseteq \text{Im } \psi \quad [\text{Claim}]$$

Other direction left to reader
 by dimension count.

(10) Thus Ringel Reps is a short exact proj. resolution for any quiver rep V .

\Rightarrow the short exact sequence

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow V \rightarrow 0$$

($n=1$) from proj. res.

$\text{Hom}_Q(-, W)$ is a left-exact contravariant functor leads to the exact sequence

$$0 \rightarrow \text{Hom}_Q(V, W) \rightarrow \text{Hom}_Q(P_0, W) \xrightarrow{\varphi} \text{Hom}_Q(P_1, W)$$

Rightmost map not nec. surjective.

$$\text{cokernel } \text{Hom}_Q(P_1, W) / \text{Im } \varphi$$

is denoted as $\text{Ext}_Q^1(V, W) = \text{Ext}_Q^1(V, W)$

Fact: $\text{Ext}_Q^i(V, W)$ does not depend on the choice of projective res.

In the category of quiver reps,

$\text{Ext}_Q^i(V, W)$ vanish for $i \geq 2$.

Homological algebra fact:

This definition of $\text{Ext}_Q^1(N, M)$ also agrees with the one from page 5. (Proven using snake lemma and Long Exact Sequence involving Hom and Ext .)

(11) Def of $\text{Ext}^i(V, W)$ for general modules.

Let V be an A -module, and

$$\cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{\dots} P_0 \xrightarrow{\varepsilon} V \rightarrow 0$$

be a projective resolution.

For A -module W , we get

$$\begin{aligned} 0 \rightarrow \text{Hom}_A(V, W) &\xrightarrow{\varepsilon^*} \text{Hom}_A(P_0, W) \xrightarrow{d_0^*} \\ \text{Hom}_A(P_1, W) &\xrightarrow{d_1^*} \cdots \xrightarrow{d_{n-1}^*} \text{Hom}_A(P_{n-1}, W) \\ &\xrightarrow{d_n^*} \text{Hom}_A(P_n, W) \xrightarrow{d_{n+1}^*} \cdots \end{aligned}$$

$$\text{Def: } \text{Ext}_A^i(V, W) = \ker d_{i+1}^* / \text{im } d_i^*$$

$$\text{with } \text{Ext}_A^0(V, W) = \ker d_0^* \cong \text{Hom}_A(V, W).$$

The fact that $\text{Ext}_Q^i(V, W) = 0$ for $i \geq 2$ is the non-trivial statement that the path algebra $A = kQ$ is hereditary, otherwise known as of global dim ≤ 1 .

Any kQ -module M has a proj. resolution of length one $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$.

Called hereditary because if kQ -module P is projective, then any submodule $M \subseteq P$ is also projective. (if Q has no oriented cycles)

(12) Claim: If P is projective, then
for any module M , $\text{Ext}^1(P, M) = 0$,

(in fact, if N is not proj., $\text{Ext}^1(N, M) \neq 0$
for some M .)

$0 \rightarrow P_1 \xrightarrow{\cong} P \rightarrow 0$ is an exact
induces P sequence

$$0 \rightarrow \text{Hom}(P, M) \xrightarrow{\cong} \text{Hom}(P_1, M)$$

and $\text{coker } \varphi = 0$ ✓

This agrees with the pg. 5 def

$$\text{Ext}^1(P, M) = \{ \text{extensions up to equiv} \}$$

Since P is projective, any exact
sequence

$$0 \rightarrow M \hookrightarrow L \xrightarrow{\alpha} P \rightarrow 0$$

is split exact, meaning there exists

$$B: P \rightarrow L \text{ s.t. } \alpha B = \text{id}_P = i_L \Rightarrow L \cong M \oplus P.$$

So up to equivalence, only one choice
for L in short exact sequence.

We use this to prove heredity:

PF: Let $M \subseteq P$ with P projective.

$$\text{Ext}^1(M, N) \cong \text{Ext}^2(P/M, N) = 0 \forall N$$

by above claim/note, $\Rightarrow M$ proj.

Prop:

(13) Let V, W be \mathbb{Q} -reps
(\mathbb{Q} has no oriented cycles). Then

$$\dim_{\mathbb{Q}} \text{Hom}_{\mathbb{Q}}(V, W) - \dim_{\mathbb{Q}} \text{Ext}^1_{\mathbb{Q}}(V, W) \\ = \langle \underline{\dim} V, \underline{\dim} W \rangle.$$

In particular, this difference doesn't depend on the choices of V, W , only on their dim vectors.

Pf: Apply $\text{Hom}(-, W)$ to the Ringel Resolution

$$0 \rightarrow \text{Hom}(V, W) \rightarrow \text{Hom}\left(\bigoplus_{i \in Q_0} kQe_i \otimes e_i V, W\right) \\ \rightarrow \text{Hom}\left(\bigoplus_{a \in Q_1} kQe_{ta} \otimes e_{sa} V, W\right) \\ \rightarrow \text{Ext}^1(V, W) \rightarrow 0.$$

$$\dim \text{Hom}(kQe_i \otimes e_j V, W) \\ = (\dim e_j V) \cdot (\dim \text{Hom}(kQe_i, W)) \\ = \dim V(j) \circ \dim W(i)$$

so we get alternating sum

$$\dim \text{Hom}(V, W) - \sum_{i \in Q_0} d_i^V d_i^W \\ + \sum_{a \in Q_1} d_{ta}^W d_{sa}^V - \dim \text{Ext}^1(V, W) = 0$$

(Actually this is a non-symm. version) \blacksquare