

# Lecture 24: Cluster Algebras and Auslander-Reiten Theory

Note Title

Gregg Musiker 8680

(4/18/2011)

① Let  $Q$  be an acyclic quiver.

Today we construct the cluster category associated to such  $Q$ .

First, some definitions:

Let  $\mathcal{D} = \mathcal{D}^b(\text{mod } kQ)$  be the derived category of bounded complexes.

In particular, objects of  $\mathcal{D}^b$  are complexes of  $kQ$ -modules with zeros on either end:

$$\dots \rightarrow 0 \rightarrow \dots \rightarrow M^p \xrightarrow{d^p} M^{p+1} \rightarrow \dots \rightarrow 0 \rightarrow \dots$$

In general, derived categories are very complicated, but since  $kQ$  is a hereditary algebra,

[Recall:  $\text{Ext}^i(M, N) = 0$  for  $i \geq 2$   
if  $M, N$  are  $kQ$ -modules]

We have the following simpler description of  $\mathcal{D}$ :

Indecomposables of  $\mathcal{D}$  are of the form  $M[i]$   
where  $M$  is an indecomp.  $kQ$ -module  
and  $i \in \mathbb{Z}$  is the shift in  $\mathcal{D}$

$$kQ \quad \mathcal{D} \quad M[0] \\ \Downarrow \quad \hookrightarrow \dots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \dots$$

② If  $M[0] = \dots \rightarrow 0 \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \dots$

$M[1] = \dots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow 0 \rightarrow \dots$

left-ward shift.

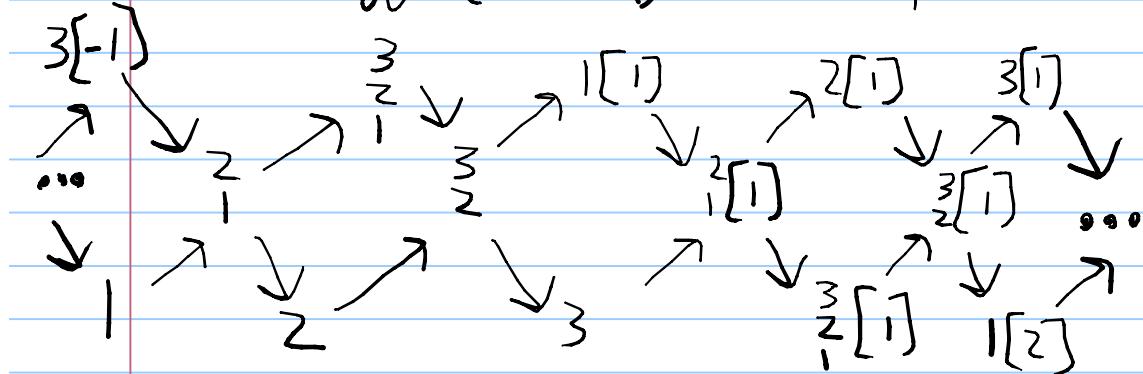
### Morphisms of $\mathcal{D}$

$$\text{Hom}_{\mathcal{D}}(M[i], N[j]) = \text{Ext}_{kQ}^{j-i}(M, N)$$

$$= \begin{cases} \text{Hom}_{kQ}(M, N) & \text{if } j = i \\ \text{Ext}_{kQ}^1(M, N) & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Example:  $Q = 1 \leftarrow 2 \leftarrow 3$

Then  $\mathcal{D}^b(\text{mod } kQ)$  can be presented as



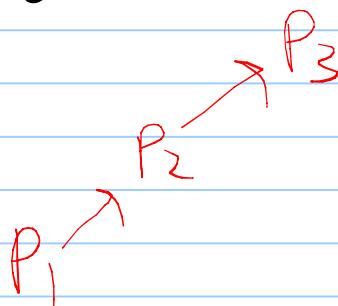
Here we have drawn maps between indecomposables of  $\mathcal{D}$ .

③ Let  $Q$  be an orientation of a Dynkin Diagram.

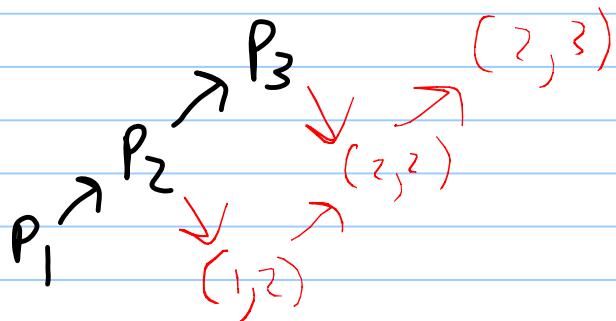
Then we can form the AR quiver of  $kQ$  by starting with

a copy of  $Q^{\text{op}}$  w/ vertices  $i$  of  $Q$  replaced with Projective indecomps  $P_i$ .

$$\text{e.g. } Q = 1 \leftarrow 2 \leftarrow 3$$



Then to the right, make a copy of  $Q^{\text{op}}$  with arrows between  $(i, k) \nleftrightarrow (j, k+1)$  if  $i \rightarrow j$  in  $Q$



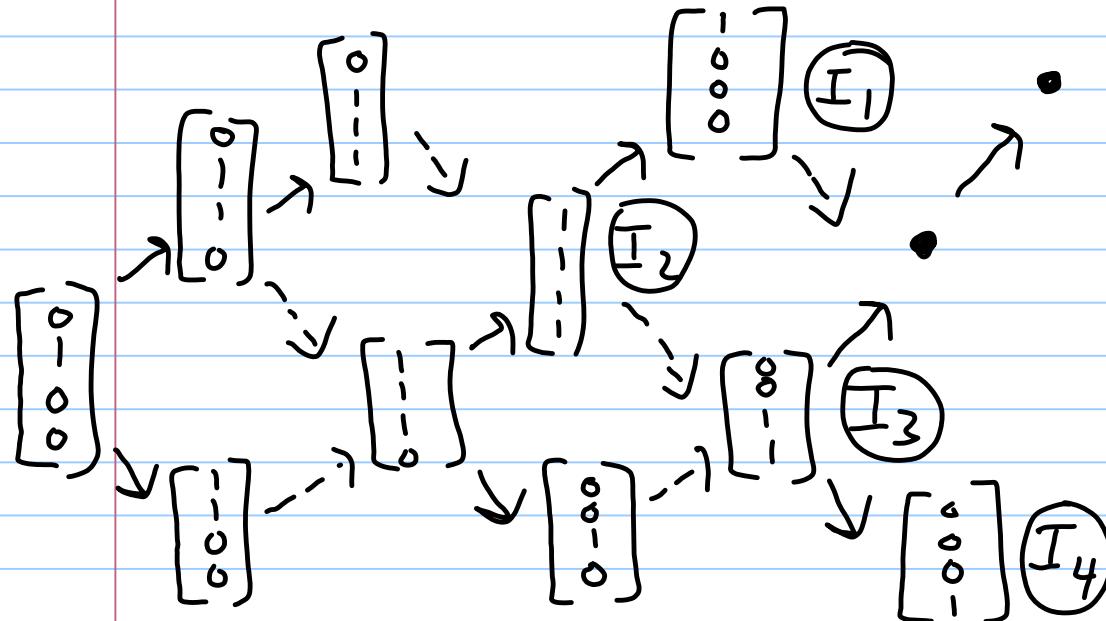
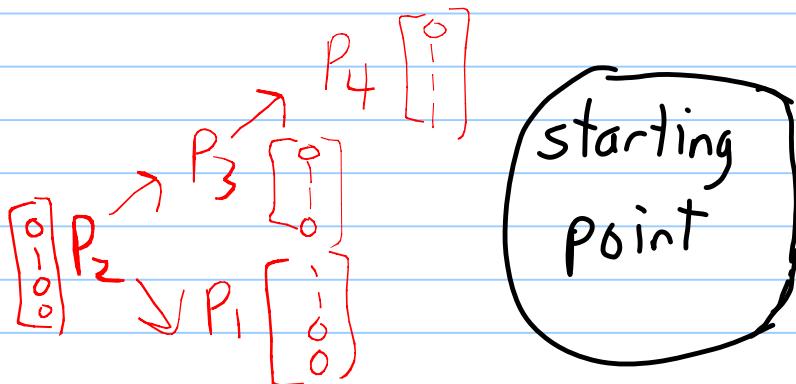
$$M(i, k+1) := \underset{\text{reflection functor to sources}}{\subset} M(i, k)$$

$$\begin{aligned} \text{in e.g., } P_1 &= k \leftarrow 0 \leftarrow 0 & M(1,2) \\ P_2 &= k \leftarrow k \leftarrow 0 & M(2,2) \\ P_3 &= k \leftarrow k \leftarrow k & M(3,2) = 0 \end{aligned}$$

$$\begin{array}{c}
 (4) \quad P_3[1] = I_1 \\
 \xrightarrow{\quad} P_2[1] \xrightarrow{\quad} I_2[0] \\
 P_1[0] \xrightarrow{\quad} E_2 \xrightarrow{\quad} E_3[0] = I_3
 \end{array}$$

Can also be obtained from  
 knitting algorithm for dim vectors  
 (since real indecomps. uniquely  
 determined by dim)

Another e.g.  $A_4 \overset{?}{\rightarrow} \overset{?}{\leftarrow} 4$



⑤ Ending point w/ Injectives

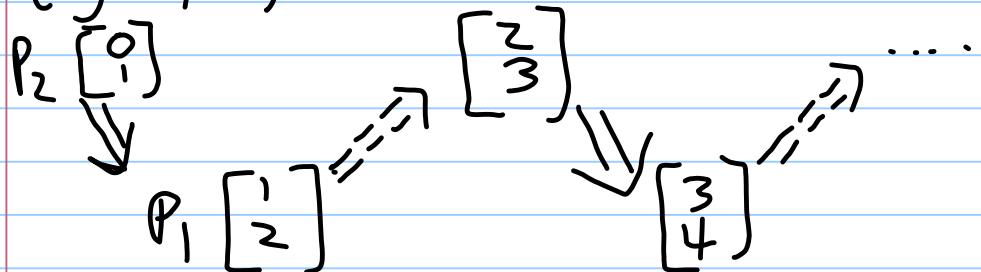
$$\dim I_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, I_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, I_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, I_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For a general algebra, not just the path algebra, AR quiver defined by "almost split exact seqs" and the maps between indecomps are irreducible morphisms

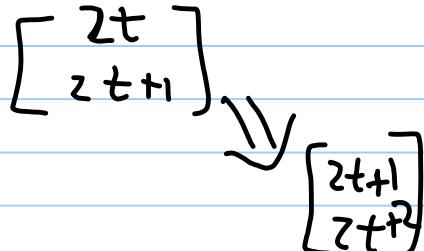
IF  $Q$  is not of finite type

AR Quiver continues infinitely

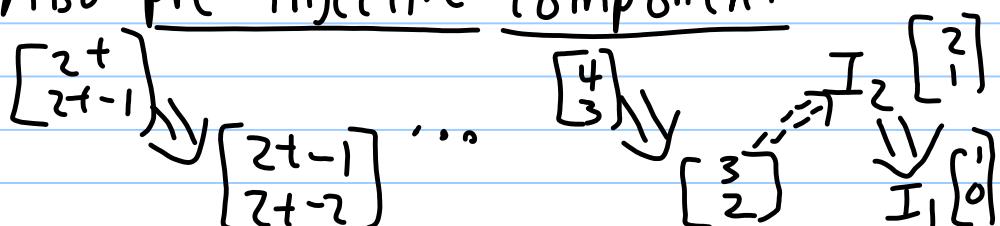
e.g.  $I = \mathbb{Z}$



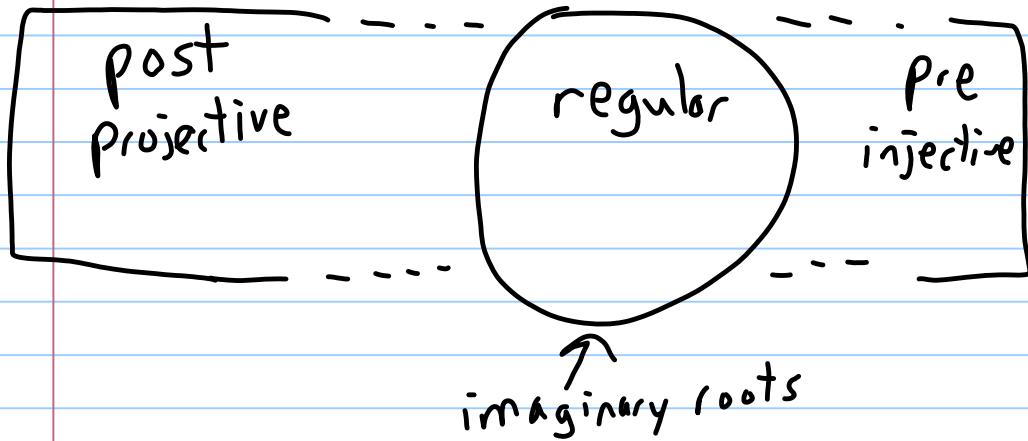
Post-projective component



Also pre-injective component



⑥ Real roots = dim vectors  
in post-projectives or pre-injectives.



Remark  $\mathcal{D}$  has Serre duality  
which means there is an equivalence  
 $\tau: \mathcal{D} \rightarrow \mathcal{D}$  called

Auslander-Reiten translation satisfying

$$\text{Hom}_{\mathcal{D}}(M, N[\cdot]) \cong D \text{Hom}_{\mathcal{D}}(N, \sim M)$$

where  $D = \text{Hom}_K(-, K)$  is  
the usual dual map.

In particular, the AR translate is

$$\sim M := D \text{Ext}^1(M, K_Q).$$

AR quiver gives way to organize rigid  
indecomposables  $\longleftrightarrow$  cluster vars

Notice: AR Quiver sits inside  
 $\mathcal{D}$  derived category.

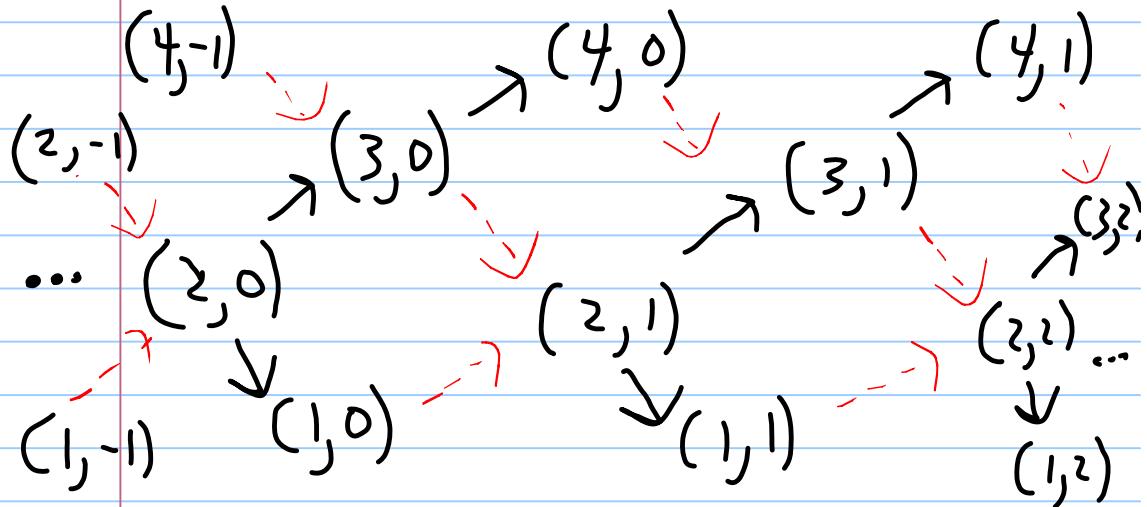
## ⑦ Back to $\mathcal{D}$ and cluster category

General construction for acyclic quivers.

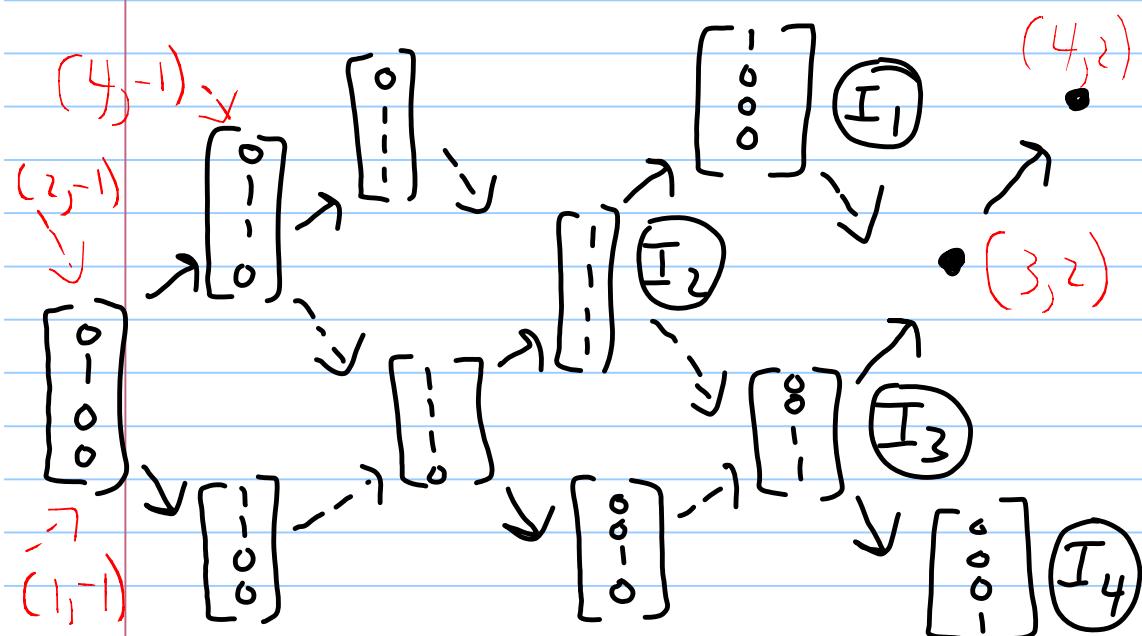
$\mathbb{Z} Q^{\text{op}}$  (Brattelli diagram)

w/  $(i, j, k) \rightarrow (j, k+1)$  if  $i \rightarrow j$  in  $Q$ :

e.g.  $1 \rightarrow 2 \leftarrow 3 \leftarrow 4$



Comparing with



⑧ If we used the knitting alg,

$$\dim M(3,2) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \dim M(1,-1) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\dim M(4,-1) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

---

$$\text{Let } F = \tilde{\tau}_{\partial}^{-1}[1].$$

Def: The cluster category

$\mathcal{C}_Q$  is the orbit category  $\mathcal{D}/F$ .

Objects = isom classes of  $F$ -orbits

Morphisms =  $\text{Hom}_{\mathcal{C}_Q}(\tilde{M}, \tilde{N})$

$$= \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(M, F^i N)$$

$\mathcal{C}_Q$  is a triangulated category

w/ AR triangles

$$L \rightarrow M \rightarrow N \rightarrow L[1]$$

Initial cluster vars  $\leftrightarrow \tilde{\tau}^{-1} P_i$   
 $= P_i[1].$