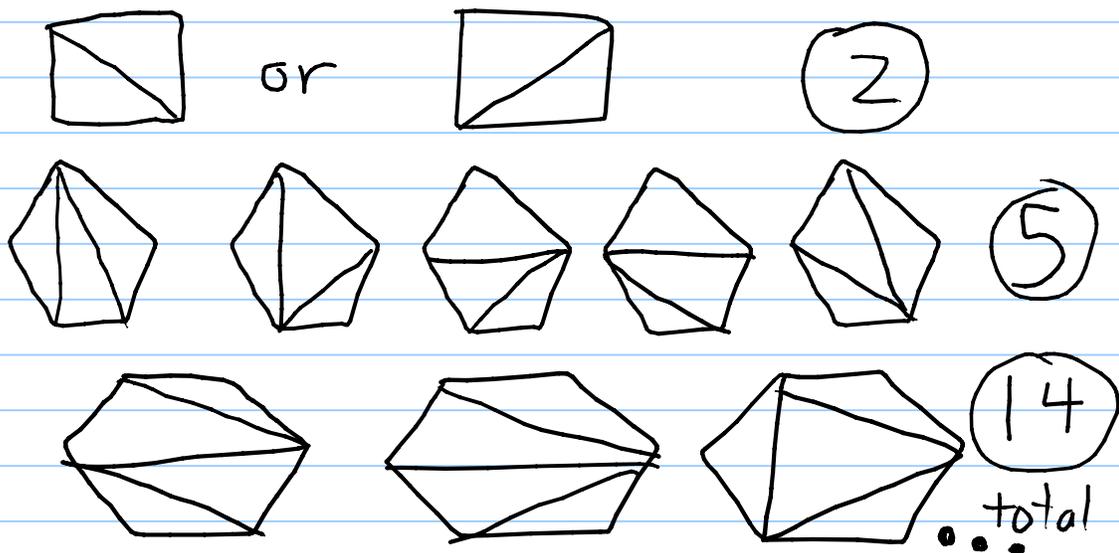


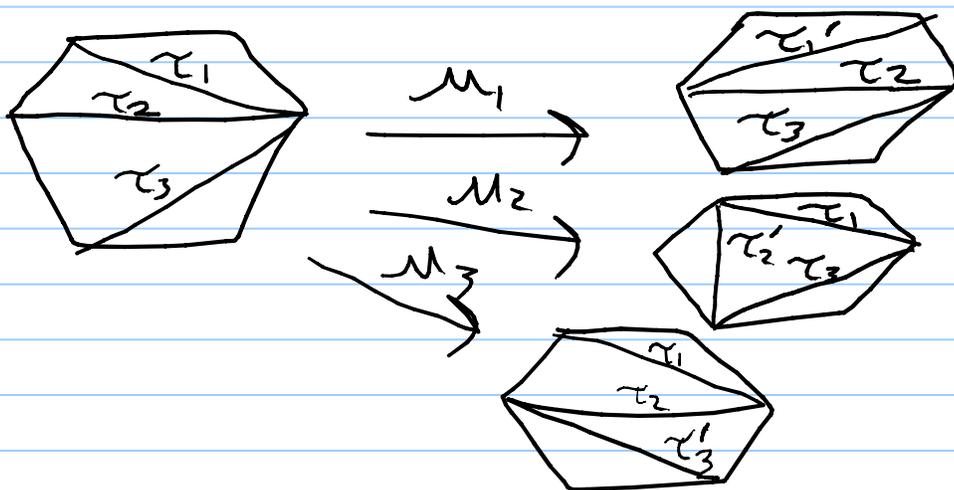
① Lecture 3: Cluster Complexes and Exchange Matrices as Quivers 1/26/11
 Math 8680 *Gregg Musiker*

Recall that the number of triangulations of an $(n+3)$ -gon is the $(n+1)$ st Catalan #

$$C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$$



Def: Given a labeled triangulation T of an $(n+3)$ -gon, we define $\mu_1, \mu_2, \dots, \mu_n$ to be the flip of arc $\tau_1, \tau_2, \dots, \tau_n$ of T .

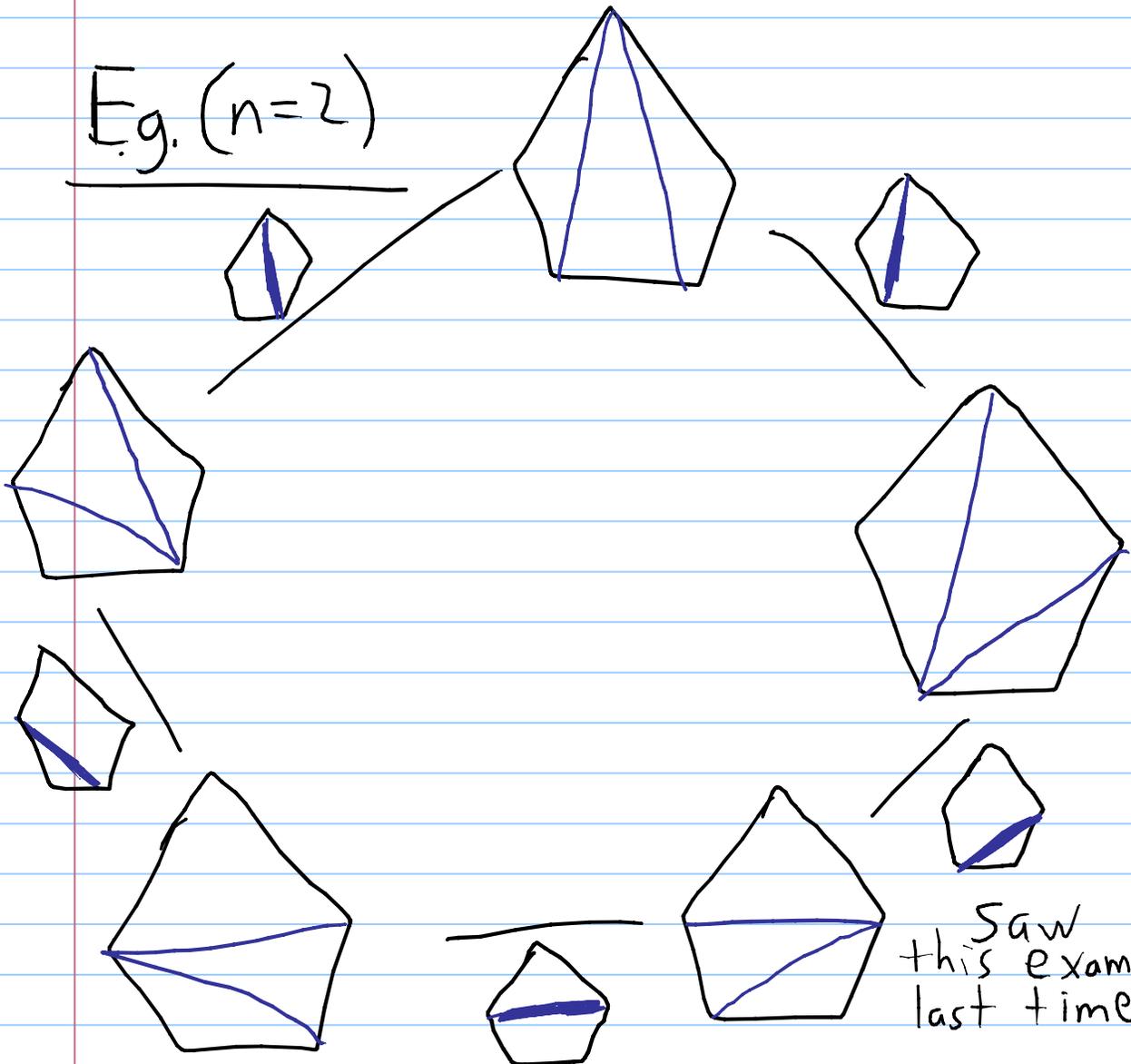


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Def: The n -associahedron is defined as the cell complex whose vertices are triangulations of an $(n+3)$ -gon and whose edges are flips.

By above, every vertex has deg n and we fill this graph out to a full n -dim complex by letting an $(n-d)$ -dim face correspond to a partial triangulation containing d arcs.

Eg. ($n=2$)



Note: All cl. algs will be of geom. type unless specified o.w.

④ Switching gears back to cluster algebras, recall we start w/ one labeled seed and mutate in n directions

This gives us the exchange graph whose vertices are all possible labeled seeds mutation-equivalent to the original one.

Def: The cluster algebra

$A = A(\tilde{X}, \tilde{B})$ defined to be

the subalgebra of $\mathbb{Z}[u_1^{\pm 1}, \dots, u_m^{\pm 1}][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

generated by

$\bigcup_{\text{all seeds}} \tilde{X}' \leftarrow$ this set called the set of

$S' \sim S = (\tilde{X}, \tilde{B})$

cluster variables

with relations of A induced by the Binomial Exchange relations.

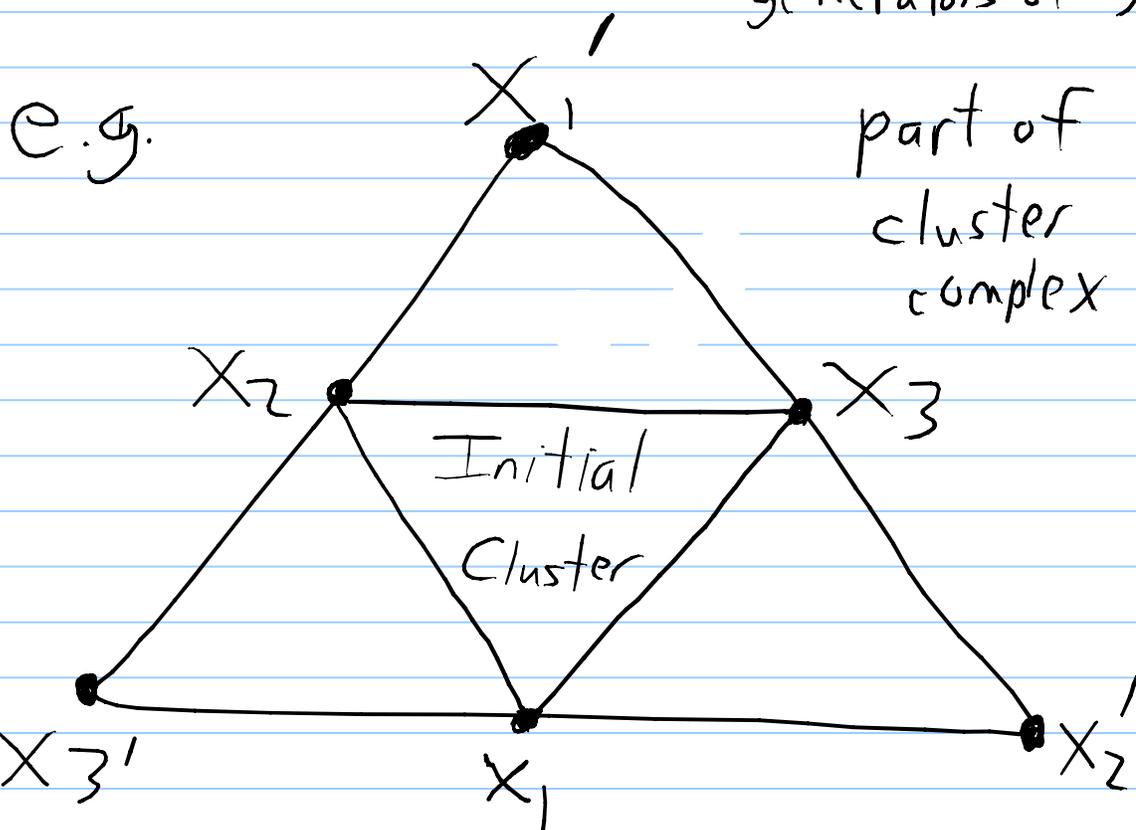
⑤ Def: The cluster complex

is defined to be the simplicial complex whose

vertices \leftrightarrow cl. vars

k -faces \leftrightarrow k -set of cl. vars that is algebraically independent

facets \leftrightarrow clusters
||
maximal sets of alg. independent generators of A

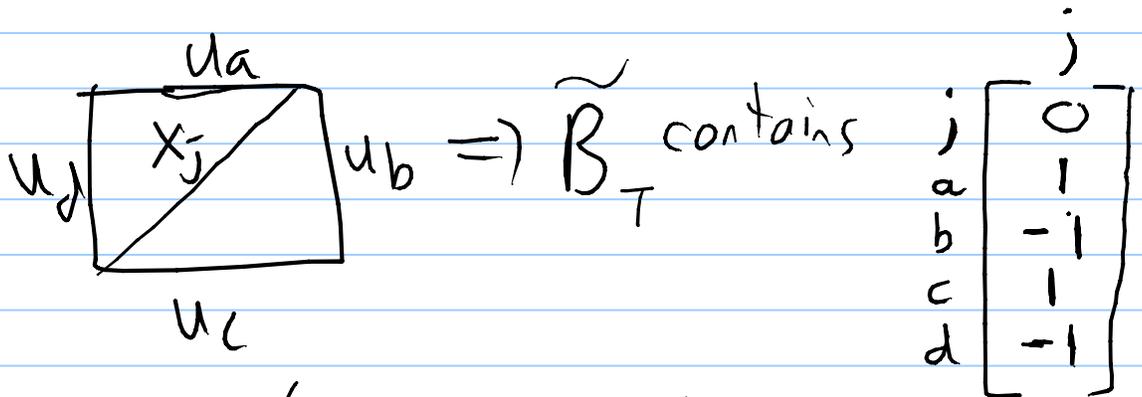


⑥

Thm [Fomin-Zelevinsky]

Given a triangulation T of an $(n+3)$ -gon such that the n chords & $(n+3)$ boundary edges

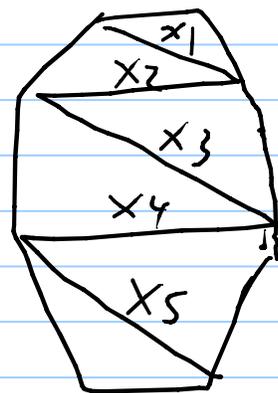
are labeled $x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_{n+3}$, define \tilde{B}_T to be the $(2n+3) \times n$ matrix whose j th column corresponds to the Ptolemy exchange associated to the j th chord.



Then $A(\{x_1, \dots, x_n, u_1, \dots, u_{n+3}\}, \tilde{B}_T)$ is a cluster algebra of rank n (of geom. type) whose cluster complex is the dual of an n -dimensional associahedron.

⑦ Such a Cluster algebra is known to be of type A_n for reasons we now explain.

Notice that if $T_0 =$



then the top $n \times n$ submatrix B_{T_0} looks like

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \text{skew-symmetrized version of } A_n - \text{Cartan matrix}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

We also now talk about how to represent a skew-symmetric B -matrix as a quiver, and will see $\text{Quiv}(B_{T_0})$ is an orientation of a type A_n Dynkin Diag.

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Def: Given skew-symmetric $n \times n$ matrix B , we define $\text{Quiver}(B)$ to be the directed graph on n vertices s.t.

there are $|b_{ij}|$ edges between v_i and v_j .

if $b_{ij} > 0$, $v_i \rightarrow v_j$
if $b_{ij} < 0$, $v_j \rightarrow v_i$

examples $\text{Quiver}(B_{T_0}) =$



$$\text{Quiver} \left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right) = \begin{array}{ccc} \bullet & \rightarrow & \bullet & \leftarrow & \bullet \\ 1 & & 2 & & 3 \end{array}$$

$$\text{Quiver} \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right) = \begin{array}{ccc} \bullet & \leftarrow & \bullet & \leftarrow & \bullet \\ 1 & & 2 & & 3 \end{array}$$

$$\text{Quiver} \left(\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \right) = \begin{array}{ccc} \bullet & \rightarrow & \bullet & \rightarrow & \bullet \\ 1 & & 2 & & 3 \end{array}$$

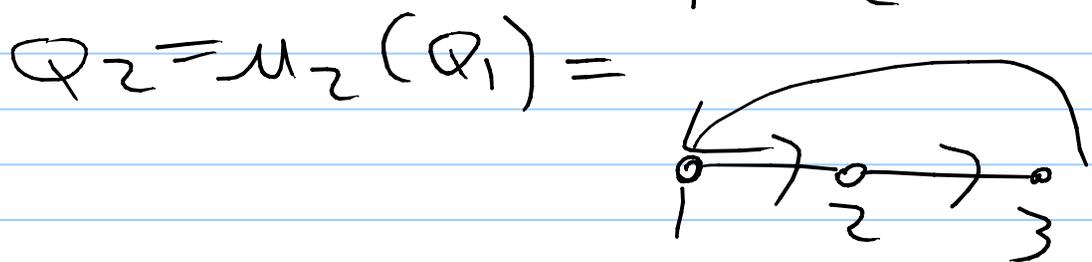
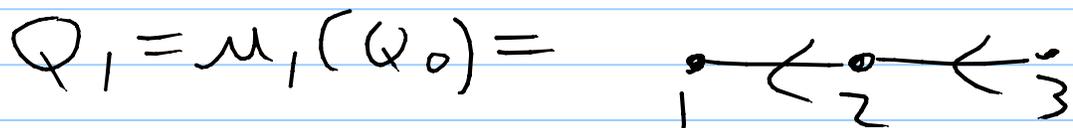
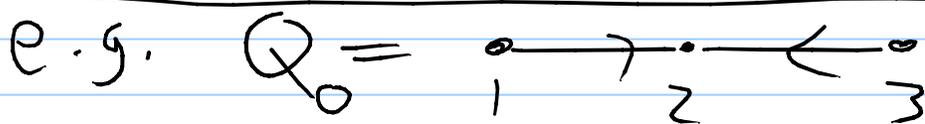
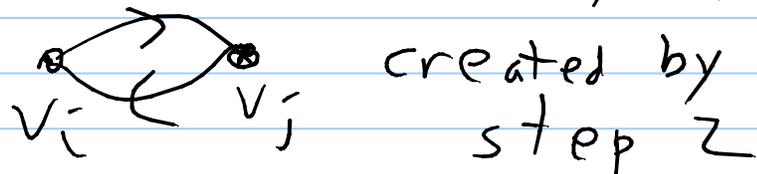
⑨ Def (Quiver mutation) Let Q be a quiver on $(n+m)$ vertices and $k \in \{1, 2, \dots, n\}$. (No loops or 2-cycles in Q)
 we define $Q' = \mu_k(Q)$ by the following 3-step process:

1) reverse the direction of all arrows incident to v_k .

2) for every 2-path



3) remove all 2-cycles



(10) We allow our quiver to have vertices v_{n+1}, \dots, v_{n+m} which cannot be mutated; called frozen.

In this way, can assign a quiver to $(m+n) \times n$ matrix also.

(No edges from one frozen vertex to another.)

Claim: Matrix Mutation
and Quiver Mutation agree
when $\tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}$ where $B = n \times n$
skew-symm.

in other words, $\mu_k(\text{Quiv}(\tilde{B}))$
 $= \text{Quiv}(\mu_k(\tilde{B}))$

Pf: Compare \tilde{B}' rule to $\mu_k(B)$ rule

$$\tilde{b}_{ij}' = \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} \pm b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} & \text{o.w.} \end{cases}$$

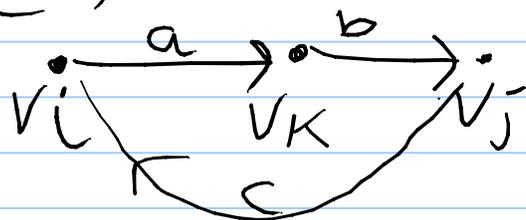
(11) Note that F-Z consider "diagrams" of skew-symmetrizable matrices

e.g. $\begin{bmatrix} 0 & z \\ -1 & 0 \end{bmatrix} \longleftrightarrow \begin{array}{c} \bullet \\ \downarrow \\ v_1 \end{array} \xrightarrow{z} v_2$

where $\Gamma(B)$ is a weighted digraph w/ $v_i \rightarrow v_j$ having weight $|b_{ij} b_{ji}|$.

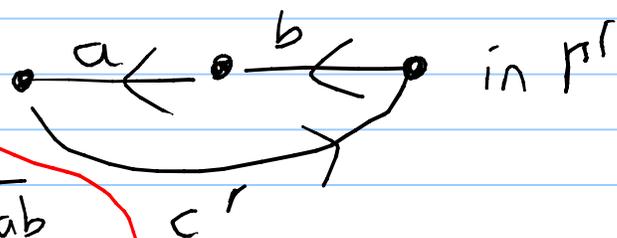
$\Gamma' = \Gamma(\mu_K(B))$ is defined the same way as quiver mutation except we have in place of (z),

(z') if in Γ we have



w/ $a, b \in \mathbb{Z}_{>0}$,
 $c \in \mathbb{Z}$ [neg c means opposite direction]

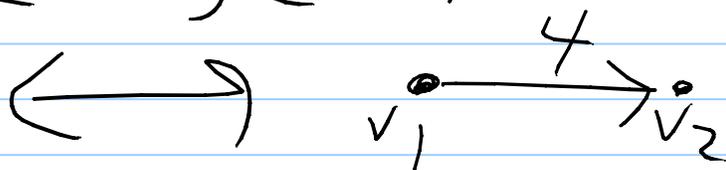
then we have



where $\sqrt{c} + \sqrt{c'} = \sqrt{ab}$

(12) Thus a seed for a skew-symm. cluster algebra can be thought of as a quiver + $\{x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}\}$ instead, and at the risk of losing some info, as a diagram instead of a quiver in non-skew-symm. case.

E.g. $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ both



Next week: we consider cluster algebras w/ seed given by quiver/diagram as orientation of

