

Lecture 4: Reflection

Groups and Coxeter Diagrams

Math 8680
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Note Title

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Today we give a crash course on root systems and finite reflection groups.

Let V be a finite dim vector space over \mathbb{R} , and let $\langle \cdot, \cdot \rangle$ denote a symmetric positive-definite bilinear form on V .

$$\langle x, y \rangle = \langle y, x \rangle$$

$$\langle x, x \rangle > 0 \text{ if } x \neq 0 \quad (\langle 0, 0 \rangle = 0)$$

$$\begin{aligned} \langle ax + by, cw + dz \rangle &= ac\langle x, w \rangle + bc\langle y, w \rangle \\ &\quad + ad\langle x, z \rangle + bd\langle y, z \rangle \end{aligned}$$

for $a, b, c, d \in \mathbb{R}$

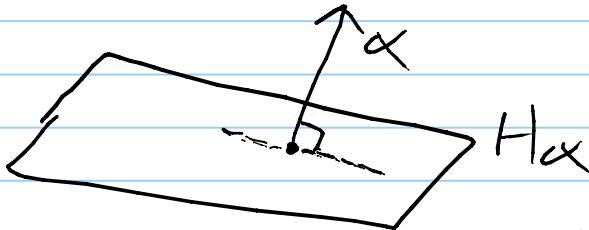
i) We can use $\langle \cdot, \cdot \rangle$ to define a norm

$$\|v\| := \sqrt{\langle v, v \rangle} \text{ for } v \in V.$$

ii) $\langle \cdot, \cdot \rangle$ can also be used to define a hyperplane:

For $\alpha \in V, \alpha \neq 0$,

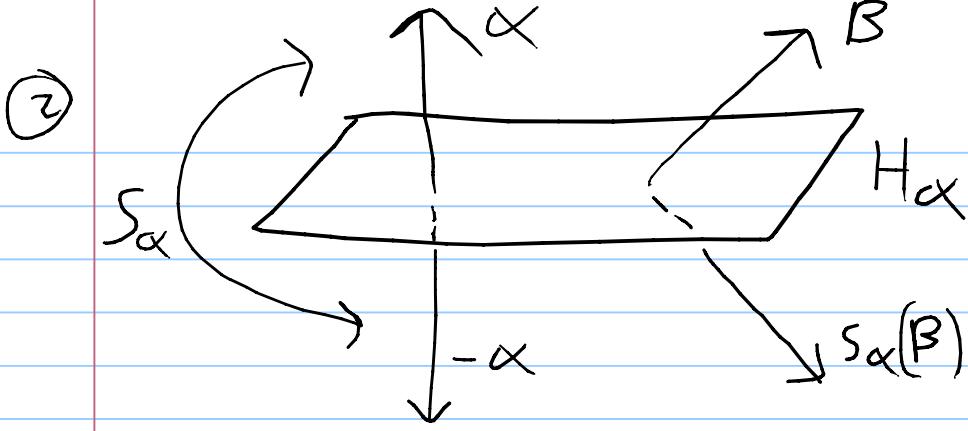
$$H_\alpha := \{ \beta \in V : \langle \beta, \alpha \rangle = 0 \}$$



Def: The reflection $s_\alpha: V \rightarrow V$ is the linear transformation on V that

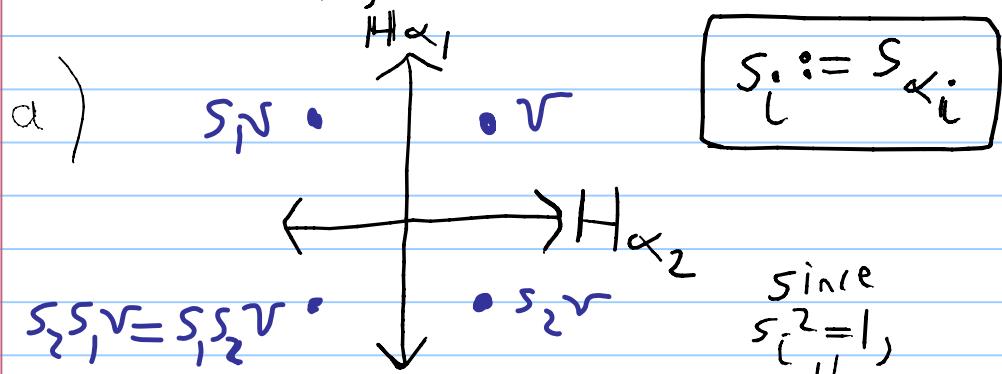
fixes H_α & sends $\alpha \mapsto -\alpha$.

$$s_\alpha(\beta) = \beta - 2 \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

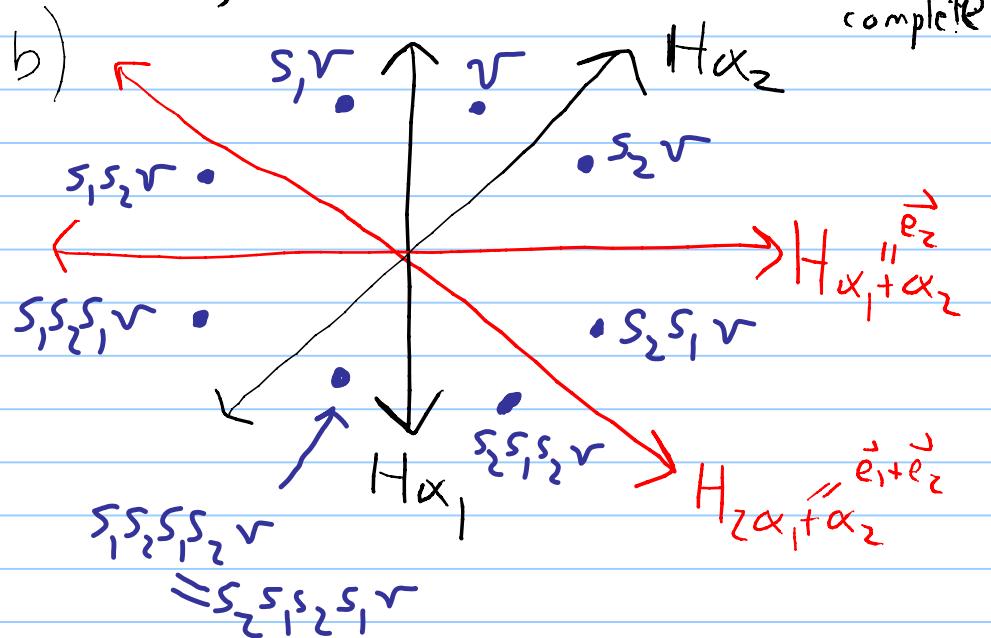


Two dimensional e.g.'s ($V = \mathbb{R}^2$)
 $\langle x, y \rangle = \text{usual } \sqrt{x^2 + y^2}$

Let $\alpha_1 = \vec{e}_1$, $\alpha_2 = \vec{e}_2$



$\alpha_1 = \vec{e}_1$, $\alpha_2 = -\vec{e}_1 + \vec{e}_2$



(3) Note that in the second example, we add in extra hyperplanes

$$H_{\alpha_1 + \alpha_2} = H_{\vec{e}_2} \text{ } \& \text{ } H_{2\alpha_1 + \alpha_2} = H_{\vec{e}_1 + \vec{e}_2}$$

to separate all the elements in the orbit of s_1 and s_2 .

This brings the notion of a finite reflection group

Def: W is a finite reflection gp on n generators if

W is generated by $\{s_1, s_2, \dots, s_n\}$ e.g. Dihedral gp includes rotations

s.t. $s_i^2 = 1$. [Note that not all elts of a reflection gp are reflections]

The above two examples further satisfied the relations

a) $s_1 s_2 = s_2 s_1$ (i.e. $(s_1 s_2)^2 = 1$)

and

b) $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$ (i.e. $(s_1 s_2)^4 = 1$),

respectively.

A group which can be modeled this way is known as a Coxeter group.

④ More on Coxeter Groups

Given an $n \times n$ symmetric matrix M , whose entries are from $\{1, 2, 3, \dots\} \cup \{\infty\}$

(with 1's only on the diagonal)
determines the Coxeter Group

$$W_M = \langle s_1, s_2, \dots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$$

$\forall 1 \leq i, j \leq n$

Note that $(s_i s_j)^\infty = 1$
is shorthand for the absence of
a relation involving a power of $s_i s_j$.

E.g.,

a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$

A Coxeter Matrix can be simply
Encoded as a Coxeter diagram

s.t. vertices $i \& j$ are connected
if and only if $m_{ij} \geq 3$

Weights of 3 are suppressed,
but weights of $m_{ij} \geq 4$ are
recorded.



⑤ Classification of finite Coxeter Gps
with two generators

$$A_1 \times A_1 \quad ; \quad \begin{matrix} s_1 \\ s_2 \end{matrix} \left\langle s_1, s_2 \mid (s_1 s_2)^2 = s_1^2 = s_2^2 = 1 \right\rangle$$

$$A_2 \quad ; \quad \begin{matrix} s_1 \\ s_2 \end{matrix} \left\langle s_1, s_2 \mid (s_1 s_2)^3 = s_1^2 = s_2^2 = 1 \right\rangle$$

$$B_2 \quad ; \quad \begin{matrix} s_1 \\ s_2 \end{matrix} \left\langle s_1, s_2 \mid (s_1 s_2)^4 = s_1^2 = s_2^2 = 1 \right\rangle$$

$$G_2 \quad ; \quad \begin{matrix} s_1 \\ s_2 \end{matrix} \left\langle s_1, s_2 \mid (s_1 s_2)^6 = s_1^2 = s_2^2 = 1 \right\rangle$$

and in general we have

$$I_2(m) = \text{Dihedral}_{2m} \quad ; \quad \begin{matrix} m \\ 1 \\ 2 \end{matrix} \left\langle s_1, s_2 \mid (s_1 s_2)^m = s_1^2 = s_2^2 = 1 \right\rangle$$

Remark: There is an important difference between Coxeter Gps

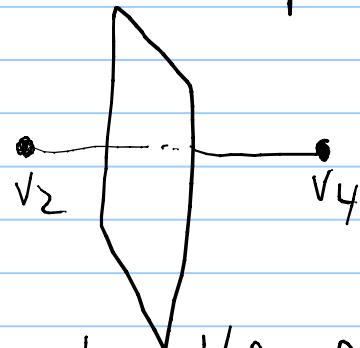
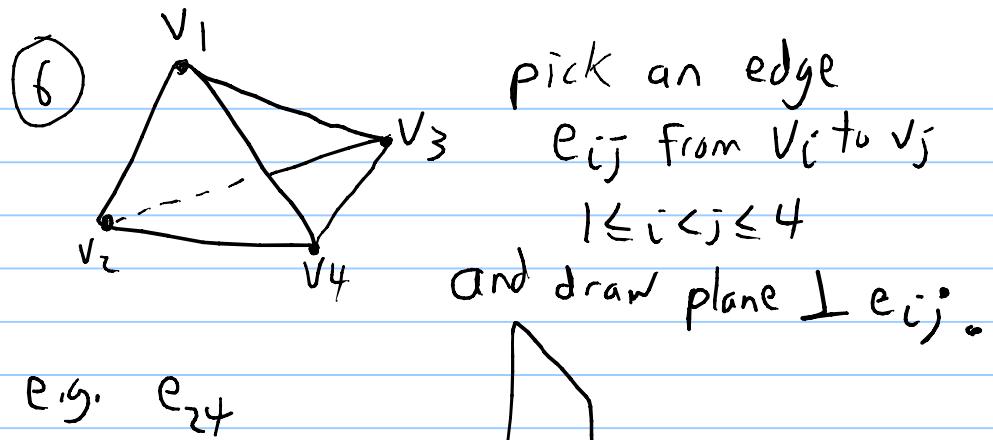
for $A_1 \times A_1, A_2, B_2, G_2$, and those

for general $I_2(m)$ [e.g. $I_2(5)$]

We will see that $I_2(s)$, in fact any $I_2(m)$ for $m \neq 2, 3, 4, 6$ is non-crystallographic.

Examples of Coxeter Groups with ≥ 3 generators :

a) Consider a regular tetrahedron in \mathbb{R}^3



Define s_{ij} to be the reflection about this plane

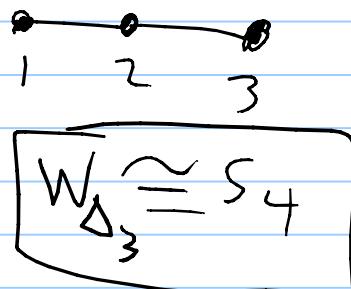
$\langle s_{12}, s_{13}, \dots, s_{34} \rangle =$ reflection group
of tetrahedron

with generating set $s_1 = s_{12}$, $s_2 = s_{13}$, $s_3 = s_{34}$

and braid relations $(s_i s_{i+1})^3 = 1$

for $1 \leq i \leq 3$,

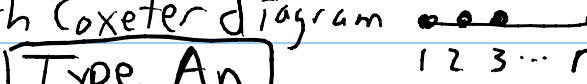
$s_i^2 = 1$ for $1 \leq i \leq 4$,



$$s_1 s_3 = s_3 s_1$$

$$s_1 s_4 = s_4 s_1$$

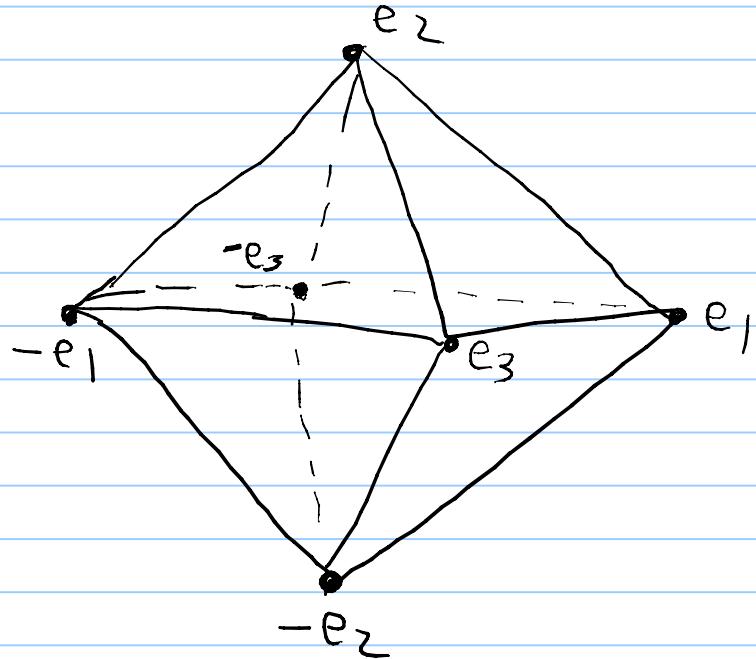
$$s_2 s_4 = s_4 s_2$$

In fact, similar analysis implies that the reflection group of the standard n -simplex,
 $W_{\Delta_n} \cong S_{n+1}$, with Coxeter diagram 

⑦ b) The n -cross-polytope is defined

as the convex hull of the endpts of
vectors $\pm e_1, \pm e_2, \dots, \pm e_n$ in \mathbb{R}^n .

Example 3-cross-polytope = octahedron



The symmetry group of the
 n -cross-polytope is B_n , the
hyperoctahedral group.

Three types of reflections:

$$\text{i) } c_1 \vec{e}_1 + \dots + c_i \vec{e}_i + \dots + c_n \vec{e}_n \quad c_i \in \mathbb{R}$$

$1 \leq i \leq n$

$$\downarrow P_i$$

$$c_1 \vec{e}_1 + \dots + (-c_i) \vec{e}_i + \dots + c_n \vec{e}_n$$

$$c_1 \vec{e}_1 + \dots + c_i \vec{e}_i + \dots + c_j \vec{e}_j + \dots + c_n \vec{e}_n$$

$$c_1 \vec{e}_1 + \dots + (c_j) \vec{e}_i + \dots + (c_j) \vec{e}_j + \dots + c_n \vec{e}_n,$$

or combo iii)

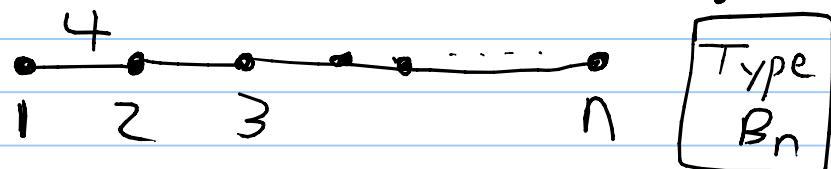
$$\xrightarrow{P_i P_j B_j} c_1 \vec{e}_1 + \dots + c_i \vec{e}_i + \dots + c_j \vec{e}_j + \dots + c_n \vec{e}_n$$

$$c_1 \vec{e}_1 + \dots + (-c_j) \vec{e}_i + \dots + (-c_j) \vec{e}_j + \dots + c_n \vec{e}_n$$

⑥ B_n is generated by $s_1 = p_{11}$ and

$$s_2 = p_{12}, s_3 = p_{23}, \dots, s_n = p_{n-1,n}$$

with associated Coxeter diagram



$$\text{since } \langle s_2, s_3, \dots, s_n \rangle \cong S_n$$

satisfying the braid relations,

$$\text{and } (s_1 s_2)^4 = 1 \begin{bmatrix} \text{equiv.} \\ p_1 p_{12} p_1 p_{12} \\ = p_2 p_1 p_{12} p_1 \end{bmatrix}$$

$$(c_1, c_2, \dots) \xrightarrow{p_1} (-c_1, c_2, \dots)$$

$$\downarrow p_{12} \quad \uparrow p_{12}$$

$$(c_2, c_1, \dots) \xrightarrow[p_1 p_{12} p_1 p_{12}]{} (-c_2, -c_1, \dots)$$

$$\uparrow p_1 \quad \downarrow p_1$$

$$(-c_2, c_1, \dots) \xrightarrow[p_1]{} (-c_2, -c_1, \dots)$$

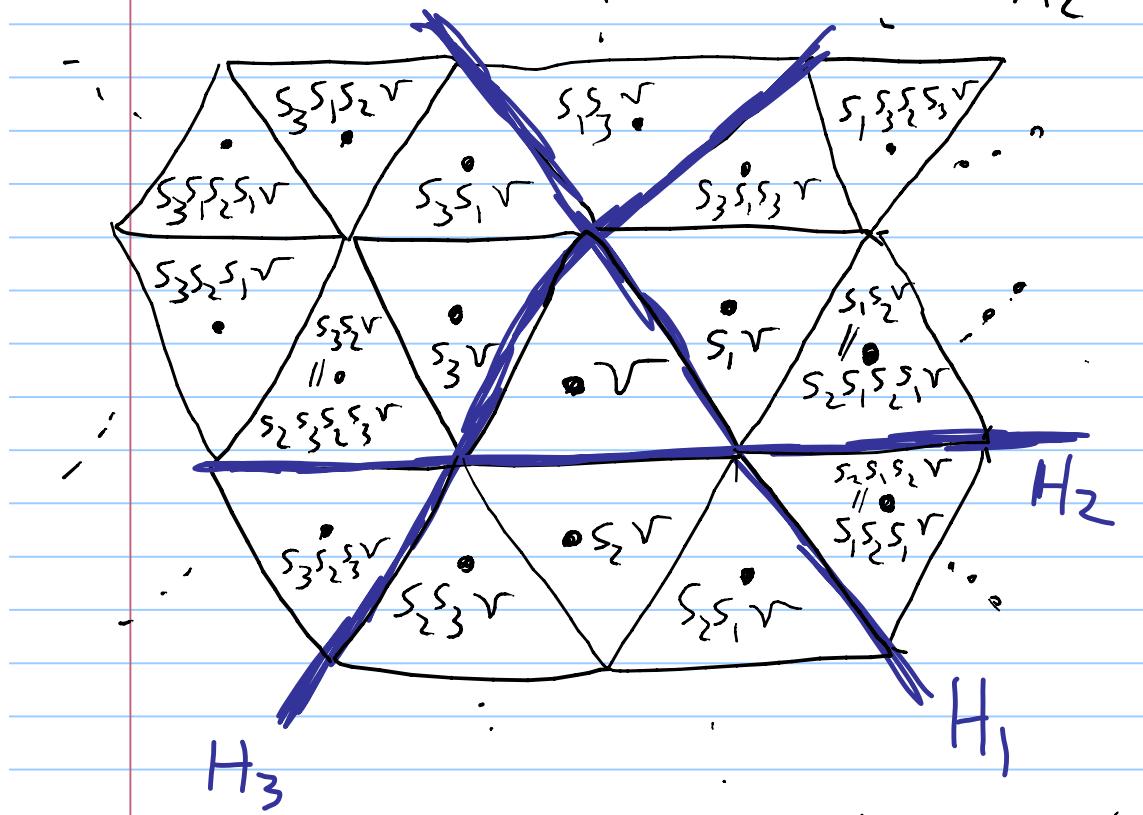
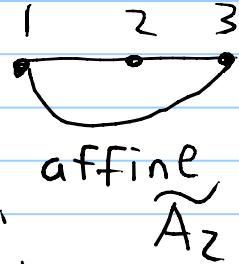
$$\downarrow p_{12} \quad \uparrow p_{12}$$

$$(c_1, -c_2, \dots) \longleftrightarrow (-c_1, -c_2, \dots)$$

⑨ Example of an infinite Coxeter Group

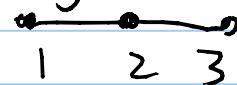
$$\left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^3 = 1 \right\rangle$$

with Coxeter diagram



Note how this e.g. contrasts w/
type A_3 w/ Coxeter diagram

no edge between



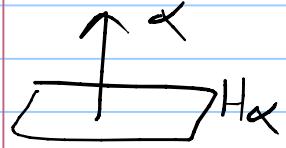
$1 \nparallel 3 \Rightarrow (s_1 s_3)^2 = 1$ in this refl. gp.
and recall from the tetrahedron / S_4 ,
 $s_1 s_3 = s_3 s_1$ ✓ since $(12) \nparallel (34)$ commute.

⑩ Root Systems : We now return
to our original set-up :

$V = \text{finite dim vec. sp. over } \mathbb{R}$
 $\langle \cdot, \cdot \rangle$ symmetric pos-definite bilinear form

We can represent reflection s_α
about hyperplane H_α , which is $\perp \alpha$,
by the formula

$$s_\alpha(\beta) = \beta - \left(\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \right) \alpha$$



 $\downarrow s_\alpha = -\alpha$

for any vectors
 $\alpha, \beta \in V, \alpha \neq 0$

Sanity checks

$$s_\alpha(\alpha) = \alpha - \frac{2\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha = -\alpha.$$

If β contained in hyperplane H_α
implies $\beta \perp \alpha \Rightarrow \langle \beta, \alpha \rangle = 0$

$s_\alpha(\beta) = \beta$ for such β 's.

Consider $V = \mathbb{R}^2, \langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \rangle = ac + bd$, std. dot product

then $s_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -b_1 \\ b_2 \end{bmatrix} \xrightarrow{\bullet \beta} s_\beta$

and $s_{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -b_2 \\ -b_1 \end{bmatrix} \xrightarrow[s_\beta]{\bullet \beta} \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$

(11) Def: A root system in V is
a subset $\Phi \subseteq V$ s.t.

- ① Φ is finite and spans V .
- ② If $\alpha \in \Phi$, then $s_\alpha(\Phi) = \Phi$.
- ③ For $\alpha \in \Phi$, $R\alpha \cap \Phi = \{\alpha, -\alpha\}$.

We will further assume the crystallographic condition
 ④ If $\alpha, \beta \in \Phi$, then $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

Lemma: If Φ is a (finite) root system, then the group generated by the reflections $\{s_\alpha : \alpha \in \Phi\}$ is finite.

Conversely, for any finite reflection group W , there is a root system s.t. $\{s_\alpha : \alpha \in \Phi\} = \{\text{reflections}\}$ in W .

Root Systems of Types A_n & B_n

$W_{A_n} \cong S_{n+1}$ with reflections

$$\xrightarrow{\text{root}} \pm(\vec{e}_j - \vec{e}_i) \quad \xrightarrow{\text{transpositions}} (i, j)$$

W_{B_n} is hyperoctahedral gp w/ reflections
 $= \{\text{transpositions}\} \cup \{(i, -i)\} \cup \{(i, -j)\}$
 $\uparrow \downarrow \uparrow$
 roots $\{\pm(\vec{e}_i - \vec{e}_j)\} \cup \{\pm \vec{e}_i\} \cup \{\pm(\vec{e}_i + \vec{e}_j)\}$
 written more simply as $\Phi_{B_n} = \{\pm e_i \pm e_j\} \cup \{\pm e_i\}$.