

2/23/15 Lecture 10: Periodic Quivers

See Section 7 of [Marsh] or [Fordy-Marsh II]

Recall from last time that certain polynomials $F(x_1, x_2, \dots, x_n)$ satisfy the Caterpillar Lemma and hence the sequence $\{x_m\}_{m \geq 1}$ defined by the recurrence

$$(*) \quad x_m x_{m+n} = F(x_{m+1}, x_{m+2}, \dots, x_{m+n}) \quad \text{are all}$$

Laurent polynomials \circ

We focus on the binomial/cluster algebra case in more depth today \circ

As mentioned last week, for certain binomials F , the Caterpillar Lemma is satisfied and not only proves Laurentness, but can be used constructively to reverse-engineer the initial cluster seed from F \circ

Today, we approach this construction from a different point of view many years later by Fordy & Marsh \circ

Rem: Let $\rho = (12 \dots n)$. If quiver Q satisfies $\mu_1(Q) = \rho Q$ [acting on vertices of Q], then

cluster variables obtained via the periodic mutation sequence $\mu_1 \mu_2 \dots \mu_n \mu_1 \mu_2 \dots$ all satisfy a recurrence of the form $(*)$ \circ

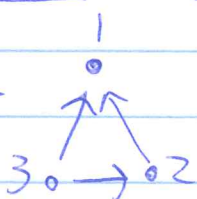
2/23/15 ② Def [Fordy-Marsh]: If $\mu_1(Q) = p(Q)$, we say that Q is a period-1 quiver.

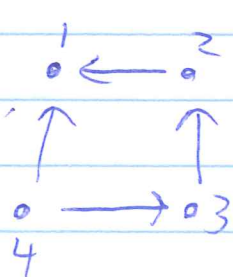
Goal: Describe all period-1 quivers.

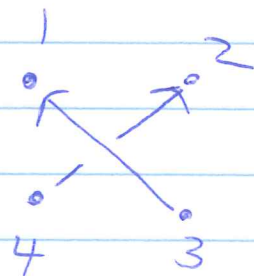
We begin with describing a two-parameter family of 1-periodic quivers that we refer to as Primitive Period 1 Quivers.

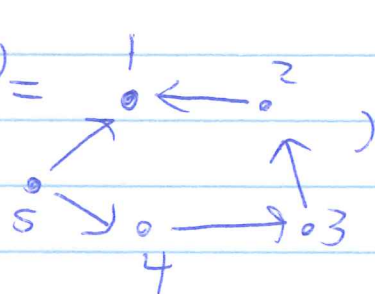
$P_n^{(k)}$ is defined as the quiver with a single edge joining vertex i and $i+k \pmod n$ for each $1 \leq i \leq n$ such that the arrow points to the smaller number.

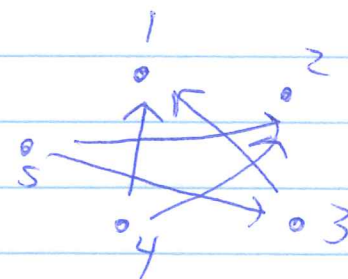
Examples: $P_2^{(1)} = 1 \leftarrow 2$ We can assume $1 \leq k \leq n/2$ without loss of generality

$P_3^{(1)} =$  $= P_3^{(2)}$

$P_4^{(1)} =$ 

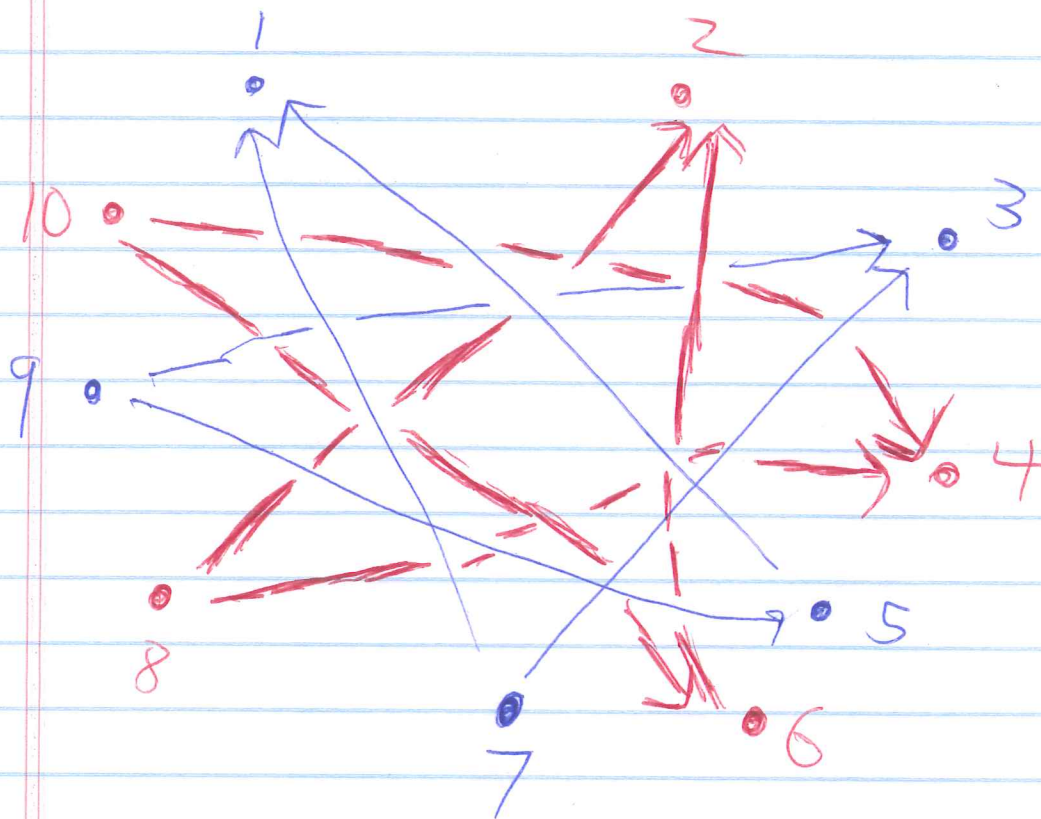
$P_4^{(2)} =$ 

$P_5^{(1)} =$ 

$P_5^{(2)} =$ 

Rem: $P_n^{(k)}$ connected $\Leftrightarrow \gcd(n, k) = 1$. If $\gcd = d$, there are d connected components, each isomorphic to $P_{n/d}^{(k/d)}$.

2/23/15 (3) Example: $P_{10}^{(4)} =$



To see this, we label the vertices of Q as $r+s$ where $1 \leq r \leq d$, $0 \leq s \leq \frac{n}{2} - 1$.

Then, $P_n^{(K)}$ contains d copies of induced subquivers $\{r, d+r, 2d+r, \dots, (n-1)d+r\}$.

No arrows between subquivers since d/k .

Lastly, $P_n^{(k)}$ connected when $\gcd(n, k) = 1$ since the residue classes (modulo n) $\{\overline{1}, \overline{1+k}, \overline{1+2k}, \dots, \overline{1+(n-1)k}\}$ are all distinct assuming relatively primeness.

2/23/15 (4) Since vertex 1 is a sink, mutating by μ , does not change the underlying undirected graph associated to Q and only results in vertex 1 being a source rather than a sink.

Since the undirected graph is also invariant under cyclic rotation, it is just as if 1 has been replaced by $n+1$ so it is treated as larger than the rest,

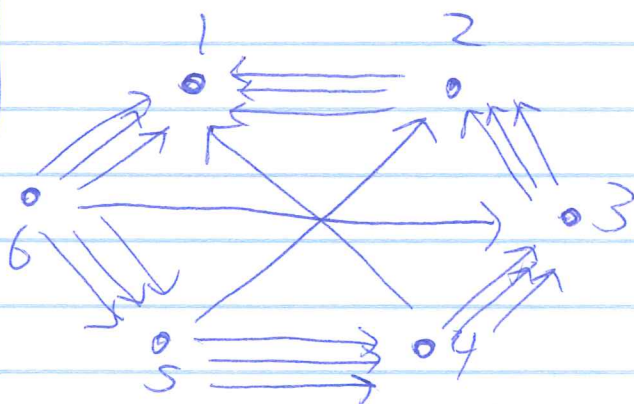
$$\mu_1 Q = Q \text{ (with vertex labels } \{2, 3, \dots, n+1\}) = \rho Q.$$

We use the primitive period-1 quivers to describe all sink-type period-1 quivers, those with vertex 1 a sink (all incident arrows are incoming).

For an n -vertex quiver and $a_1, a_2, \dots, a_{\lfloor \frac{n}{2} \rfloor} \in \mathbb{Z}_{\geq 0}$, we define $\boxed{a_1 P_n^{(1)} + \dots + a_{\lfloor \frac{n}{2} \rfloor} P_n^{(\lfloor \frac{n}{2} \rfloor)}}$ to have

a_k arrows $\bullet \xleftarrow{k} \bullet$ & rest of the arrows obtained by rotation (and reversal if needed to insure arrows point to smaller label).

Example: $\boxed{3 P_6^{(1)} + P_6^{(3)}}$

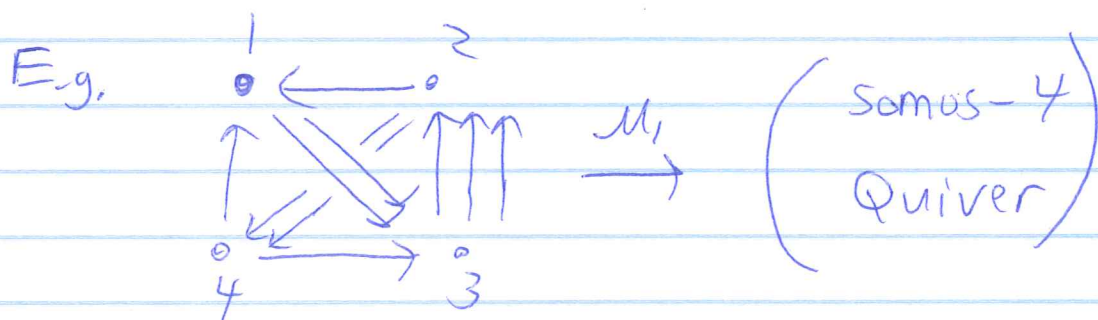


2/23/15 (5) By the same logic as above, these are all period-1 quivers as well since vertex 1 is a sink.

Moreover, by focusing on the arrows incident to vertex 1, any sink-type period 1 quiver can be written, i.e. decomposed into primitives, this way.

We now consider the more general case of period 1 quivers where vertex 1 is not nec. a sink.

Consequently, the underlying undirected graphs of $Q \neq \mu_1 Q$ are not automatically the same since we must add arrows corresponding to \mathbb{Z} -paths.



Instead, we build a general period 1 quiver from primitives by
$$a_1 P_n^{(1)} + \dots + a_{\lfloor \frac{n}{\mathbb{Z}} \rfloor} P_n^{(\lfloor \frac{n}{\mathbb{Z}} \rfloor)} + E$$

where a_i 's are now allowed to be positive or negative (or zero) and E represents a fudge factor, [subquiver on smaller # vertices]

$-k P_n^{(k)}$ represents k copies of $(P_n^{(k)})^{\circ P}$ with vertex 1 a source.

To define E , we introduce some notation.

2/23/15 (6) Given a choice of $a_1, \dots, a_{\lfloor \frac{n}{2} \rfloor} \in \mathbb{Z}$, for each $1 \leq i < j \leq \lfloor \frac{n}{2} \rfloor$

$$\varepsilon_{ij} := \frac{1}{2} (a_i |a_j| - a_j |a_i|)$$

$$= \begin{cases} \text{sgn}(a_i) \frac{a_i a_j}{\text{sgn}(a_i) |a_i a_j|} & \text{if } \text{sgn}(a_i) = -\text{sgn}(a_j) \\ 0 & \text{if } \text{sgn}(a_i) = \text{sgn}(a_j) \end{cases}$$

Let $p P_{n-2}^{(k)}$ represent the primitive on vertices $\{2, 3, \dots, n-1\}$

$p^2 P_{n-4}^{(k)}$ " " $\{3, 4, \dots, n-2\}$

$p^l P_{n-2l}^{(k)}$ " " $\{l+1, l+2, \dots, n-l\}$

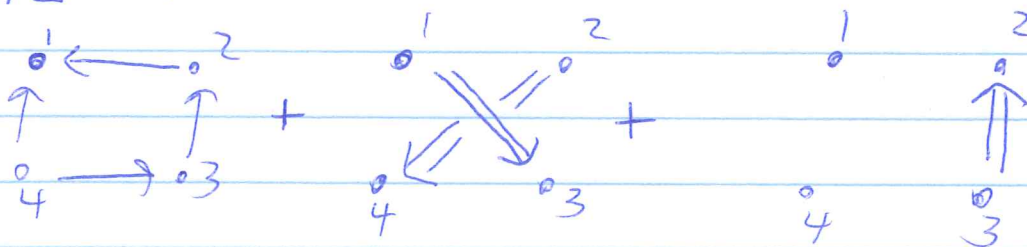
Thm 6.6 of Furdy-Marsch (7.2.1 of [Marsch])

Any period 1 quiver can be written as (on n vertices)

$$a_1 P_n^{(1)} + a_2 P_n^{(2)} + \dots + a_{\lfloor \frac{n}{2} \rfloor} P_n^{(\lfloor \frac{n}{2} \rfloor)} + \sum_{l=1}^{\lfloor \frac{n}{2} \rfloor - 1} \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor - 1} \varepsilon_{l, l+t} p^l P_{n-2l}^{(t)}$$

Example: Somos-4 quiver $= P_4^{(1)} - 2P_4^{(2)} + 2p P_{4-2,1}^{(1)}$

$$\varepsilon_{12} = \text{sgn}(-2) \cdot (1) \cdot (-2) = +2$$



2/23/15 ⑦ In particular, for every palindromic \mathbb{Z} -vector

$$[b_{12} \ b_{13} \ b_{14} \ \dots \ b_{1n}] \text{ where}$$

$$b_{1j} = (\# \text{ arrows } 1 \rightarrow j) - (\# \text{ arrows } 1 \leftarrow j)$$

There is a unique way to complete the Exchange Matrix of the corresponding period 1 quiver.

Rem: (Palindromicity required since primitives have this symmetry about vertex 1 and

$$b_{1k} = b_{1, n-k+2} = -a_k \text{ for } 2 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$$

Example: $[2 \ -1 \ 0 \ 1 \ 0 \ -1 \ 2]$

corresponds to $\underbrace{2P_8^{(1)} - P_8^{(2)} + P_8^{(4)}} + \underbrace{2P_6^{(1)} - P_4^{(2)}}$

$$\varepsilon_{12} = 2$$

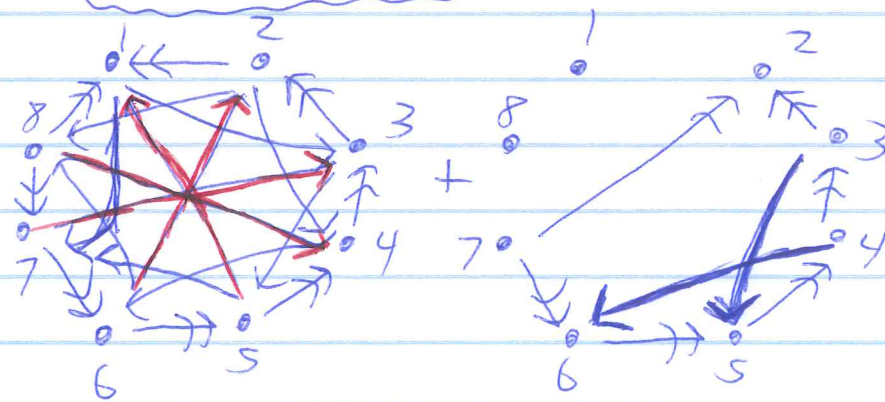
$$\varepsilon_{13} = 0$$

$$\varepsilon_{14} = 0$$

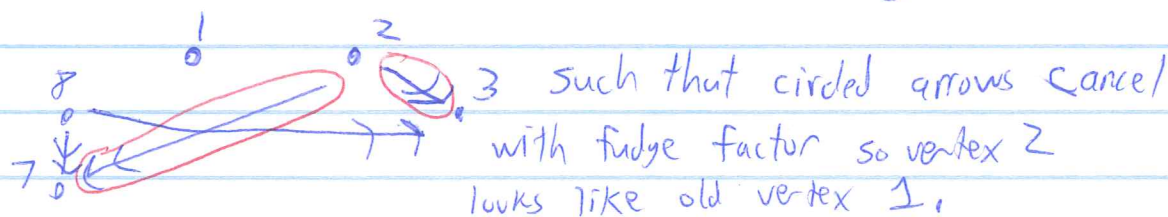
$$\varepsilon_{23} = 0$$

$$\varepsilon_{24} = -1$$

$$\varepsilon_{34} = 0$$



Notice in particular that when we apply u_1 , we obtain



2/23/15 (P) pf of Theorem: Suppose Q is a quiver with exchange matrix $B = [b_{ij}]$, i.e. $b_{ij} = (\# \text{ arrows } i \rightarrow j) - (\# \text{ arrows } i \leftarrow j)$

We wish to show

Q is 1-periodic $\Leftrightarrow \exists [m_1, \dots, m_{n-1}] \in \mathbb{Z}^{n-1}$ s.t. $m_k = m_{n-k}$ and

$$(**) \quad b_{ij} = m_{i-j} + \varepsilon_{i-j+1} + \varepsilon_{i-j+2} + \dots + \varepsilon_{j-j-1} \quad \text{for } i > j$$

Further, let $P = \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ and quiver pQ associated to exch. matrix pBp^{-1} .

Since $\begin{array}{c} \circ \longrightarrow \circ \longrightarrow \circ \\ i \qquad \qquad \qquad j \end{array}$ (difference = $i-1$)
 in Q leads to $\begin{array}{c} \circ \longrightarrow \circ \\ i \qquad \qquad \qquad j \end{array}$ (difference = $j-1$)
 in pQ

For $i > j \neq 1$, we have $[pBp^{-1}]_{ij} = b_{ij} + \varepsilon_{i-1, j-1}$

$$\left[\text{where } \varepsilon_{i-1, j-1} = \frac{1}{2} (m_{i-1} | m_{j-1} | + m_{j-1} | m_{i-1} |) \right] \quad \begin{array}{l} \text{contributions from} \\ \text{2-paths} \end{array}$$

On the other hand, $[pBp^{-1}]_{ij} = b_{i-1, j-1}$.

So Q 1-periodic $\Leftrightarrow b_{ij} + \varepsilon_{i-1, j-1} \stackrel{(*)}{=} b_{i-1, j-1} \quad \forall \quad i > j \neq 1$.

Letting $b_{i-j+1, 1} = m_{i-j}$ & iterating $(*)$ yields $(**)$ as desired.

2/23/15 (9) To see this more clearly, we note $\epsilon_{j-1, i-1} = -\epsilon_{i-1, j-1}$

$$\text{to get } b_{ij} = b_{i-1, j-1} + \epsilon_{j-1, i-1}$$

$$= b_{i-2, j-2} + \epsilon_{j-2, i-2} + \epsilon_{j-1, i-1}$$

\vdots

$$= b_{i-j+1, j} + \epsilon_{j, i-j+1} + \epsilon_{j-1, i-j+2} + \dots + \epsilon_{j-1, i-1}$$

$$= m_{i-j} + \dots = (**)$$

Analogous algebra yields Palindromicity of $[m_1, \dots, m_{n-1}]$ but this also comes from the fact we could mutate at vertex n instead to run the same recurrence backwards so we must have $m_k \leftrightarrow m_{n-k}$ symmetry.

Relation between period 1 quivers and physics

[Benvenuti-Hanany-Kazakopoulos] "The Toric Phases of the $Y^{p,q}$ Quivers" (arXIV: 0412279)

Examples of $N=1$ superconformal gauge theories.

Specific Examples of Quiver Gauge Theories

Countably infinite family of explicit non-homogeneous 5-dimensional Sasaki-Einstein metrics found by Gaiotto + Martelli, Sparks, Waldram.

2/23/15 (10) Corresponding manifolds called $Y^{p,q}$ ($q < p$ pos. integers)

Using the AdS/CFT duality (mentioned in Lecture 1)

associated dual quantum field theories are quiver gauge theories also labeled by $Y^{p,q}$.

Metric on $Y^{p,q}$ satisfies

$$(ds)^2 = \frac{1-y}{6} (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$+ \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9} (d\psi - \cos \theta d\phi)^2$$

$$+ w(y) \left[d\alpha + f(y) (d\psi - \cos \theta d\phi) \right]^2$$

$$\text{with } w(y) = \frac{2(b-y^2)}{1-y}, \quad q(y) = \frac{b-3y^2+2y^3}{b-y^2},$$

$$f(y) = \frac{b-2y+y^2}{6(b-y^2)}$$

coordinate y satisfies $y_1 \leq y \leq y_2$ where y_1, y_2 are two smallest roots of $b - 3x^2 + 2x^3 = 0$

$$\text{Parameter } b = \frac{1}{2} - \frac{(p^2 - 3q^2)}{4p^3} \sqrt{4p^2 - 3q^2}$$

Topology is $S^2 \times S^3$.

2/23/15 (11) Via AdS/CFT correspondence, corresp. quiver gauge theory can be built as a quiver on $2p$ vertices

$$P_{2p}^{(p-q)} = \sum P_{2p}^p + \underbrace{\sum p^q P_{2(p-q)}}_{\text{fudge term}} \quad [q < p]$$

In fact this is one such "toric phase" of the quiver gauge theory if $\gcd(p, q) = 1$.

Higher period if $\gcd(p, q) > 1$ the way physicists construct $Y^{p,q}$.

Related to another quiver Gauge theory $\begin{matrix} a, b, c \\ \swarrow \quad \searrow \\ p-q, p+q, p \end{matrix}$
for which $Y^{p,q}$ is special case

More on these later in course.

See R. Eager "Non-commutative geometry and brane tilings".