-/15 Lecture 16-17: Triangular Toric Diagrams, Abelian Orbifolds and the 3-dimensional McKay Correspondence

(Section 5 of Ueda-Yamazaki, "A note on dimer models and McKay quivers" arXiv:0605780)

Let G be a finite subgroup of $Gl_n(\mathbb{C})$.

Let $\{\chi_1, \chi_2, \dots, \chi_m\}$ be the irreducible representations of G.

Since G is a subgroup of $Gl_n(\mathbb{C})$, G can be presented as a group of $n \times n$ matrices.

Called the natural representation. (denote as χ_{nat})

Def (McKay quiver of G)

- vertices given by $\{\chi_1, \dots, \chi_m\}$
- for each pair χ_i, χ_j, the number of arrows $\chi_i \Longrightarrow \chi_j$ equals a_{ij} where

$$\boxed{\chi_i \otimes \chi_{nat} = \bigoplus_{j=1}^{m} a_{ij} \chi_j}$$

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Warm-up: $G = \mathbb{Z}_m \subseteq Gl_2(\mathbb{C})$

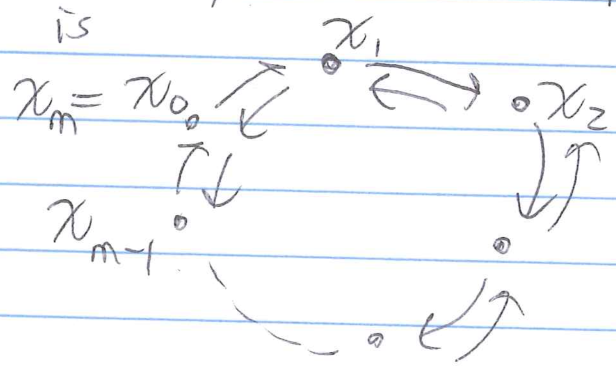
$\chi_{nat}(\sigma) = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$ and $\mathbb{Z}_m = \langle \sigma \rangle$
$\zeta = e^{2\pi i/m}$

Since $\mathbb{Z}_m$ abelian, each irred rep'n 1-dimensional

$\chi_i(\sigma) = [\zeta^i]$

$\chi_i \otimes \chi_{nat}(\sigma) = \begin{bmatrix} \zeta^{i+1} & 0 \\ 0 & \zeta^{i-1} \end{bmatrix} = \chi_{i+1} \oplus \chi_{i-1}$ (where subscripts taken mod m)

Thus McKay quiver for $\mathbb{Z}_m$ (two-dimensional) is



[arrows $i \to i+1$ and $i \to i-1$ for each vertex]

Two-dimensional McKay correspondence

$\left\{ \begin{matrix} \text{Finite subgroups} \\ G \text{ of } SL_2(\mathbb{C}) \end{matrix} \right\} \xrightarrow[\text{quiver}]{\text{McKay}} \left\{ \begin{matrix} \text{Dynkin Diagrams} \\ \text{of affine } A_n, D_n, E_6, E_7, E_8 \\ \text{as quivers} \end{matrix} \right\}$

$\bullet \rightleftarrows \bullet$ instead of $\bullet\!-\!\!-\!\bullet$

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We now consider a 3-dim McKay correspondence but focus on finite abelian subgroups of $SL_3(\mathbb{C})$.

Up to conjugation, a finite abelian subgroup $G < SL_3(\mathbb{C})$ can be presented as $\langle \sigma_1 \rangle$ or $\langle \sigma_1, \sigma_2 \rangle$ where $\sigma_i = \begin{bmatrix} \zeta^a & & \\ & \zeta^b & \\ & & \zeta^c \end{bmatrix}$ $\zeta = e^{2\pi i/m}$

$a+b+c \equiv 0 \mod m$

e.g. let $G = \mathbb{Z}_2 = \{1, \varepsilon\}$ with

$\chi_{nat}(\varepsilon) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Rem: need exactly two (-1)'s so that $\det = +1$

(i.e. $a=1, b=1, c=0, \; 1+1 \equiv 0 \mod 2$)

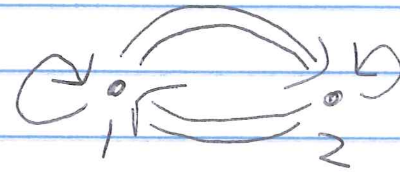
$\chi_1(\varepsilon) = [-1]$, $\chi_2 = \chi_0(\varepsilon) = [1]$ are the 2 irred rep'ns of $\mathbb{Z}_2$.

$\chi_1 \otimes \chi_{nat} = \chi_2 \oplus \chi_2 \oplus \chi_1 \left(\begin{bmatrix} (-1)^2 & & \\ & (-1)^2 & \\ & & (-1) \end{bmatrix} \text{ on } \varepsilon \right)$

χ_0 = triv rep

$\chi_{nat} = \chi_2 \otimes \chi_{nat} = \chi_1 \oplus \chi_1 \oplus \chi_2$

Thus the McKay quiver for $\mathbb{Z}_2$ is



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e.g. $G=\mathbb{Z}_3=\langle\sigma\rangle$ with $\chi_{nat}(\sigma)=\begin{bmatrix}\zeta&0&0\\0&\zeta&0\\0&0&\zeta\end{bmatrix}$

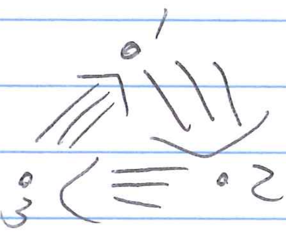
$\left(\zeta=e^{2\pi i/3}=\dfrac{-1+\sqrt{-3}}{2}\right)$

$\underset{\overset{\|}{\chi_3}}{\chi_0}\otimes\chi_{nat}=\chi_{nat}=\chi_1\oplus\chi_1\oplus\chi_1$

$\chi_1\otimes\chi_{nat}=\chi_2\oplus\chi_2\oplus\chi_2$

$\chi_2\otimes\chi_{nat}=\chi_3\oplus\chi_3\oplus\chi_3$

$\Rightarrow$ McKay quiver for $\mathbb{Z}_3$ is



Rem: we should technically say this is the McKay quiver for $\mathbb{Z}_3(1,1,1)$ ← Miles Reed notation

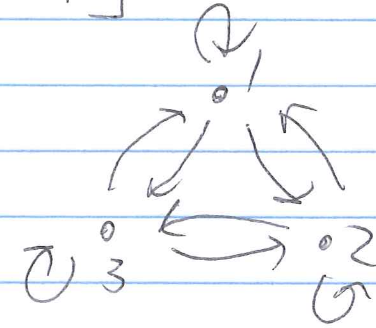
to distinguish from

$\mathbb{Z}_3(2,1,0)$ with $\chi_{nat}(\sigma)=\begin{bmatrix}\zeta^2&&\\&\zeta&\\&&1\end{bmatrix}$

with $\underset{\overset{\|}{\chi_3}}{\chi_0}\otimes\chi_{nat}=\chi_2\oplus\chi_1\oplus\chi_3$

$\chi_1\otimes\chi_{nat}=\chi_3\oplus\chi_2\oplus\chi_1$

$\chi_2\otimes\chi_{nat}=\chi_1\oplus\chi_3\oplus\chi_2$



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e.g., to contrast with the previous $\mathbb{Z}_m$ cases, we next consider $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \sigma, \tau, \sigma\tau\}$ with

$$\chi_{nat}(\sigma) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \chi_{nat}(\tau) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

note that this choice implies $\chi_{nat}(\sigma\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\chi_{11}(\sigma) = -1, \ \chi_{11}(\tau) = -1, \ \chi_{11}(\sigma\tau) = 1$

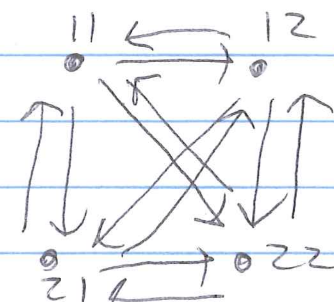
$\chi_{12}(\sigma) = -1, \ \chi_{12}(\tau) = 1, \ \chi_{12}(\sigma\tau) = -1$ $\quad \chi_{ij}(1) = +1$

$\chi_{21}(\sigma) = 1, \ \chi_{21}(\tau) = -1, \ \chi_{21}(\sigma\tau) = -1$ $\quad$ for all $i,j \in \{1,2\}$

$\chi_{triv} = \chi_{22}(\sigma) = 1, \ \chi_{22}(\tau) = 1, \ \chi_{22}(\sigma\tau) = 1$

$\chi_{11} \otimes \chi_{nat} = \chi_{22} \oplus \chi_{21} \oplus \chi_{12}$

$\chi_{12} \otimes \chi_{nat} = \chi_{21} \oplus \chi_{22} \oplus \chi_{11}$



$\chi_{21} \otimes \chi_{nat} = \chi_{12} \oplus \chi_{11} \oplus \chi_{22}$

$\chi_{22} \otimes \chi_{nat} = \chi_{11} \oplus \chi_{12} \oplus \chi_{21}$

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We next describe how to construct a potential associated with a given McKay quiver Q_G. (in the G finite abelian in $SL_3(\mathbb{C})$ case)

χ_{nat} always decomposes as a direct sum of three 1-dim irred's

$(\chi_{nat} = \chi_a \oplus \chi_b \oplus \chi_c)$ [a, b, c not nec. distinct]

Since $\chi_1, \chi_2, \ldots, \chi_m$ are all 1-dim,

$\chi_i \otimes \chi_a$, $\chi_i \otimes \chi_b$, and $\chi_i \otimes \chi_c$ are each again 1-dim irreds.

Call them $\chi_{j_{a,i}}$, $\chi_{j_{b,i}}$, and $\chi_{j_{c,i}}$

For all vertices i in Q_G

Let X_i denote the arrow $\chi_i \to \chi_{j_{a,i}}$

Y_i " " $\chi_i \to \chi_{j_{b,i}}$

Z_i " " $\chi_i \to \chi_{j_{c,i}}$

Define potential W_G as

$$\left(\sum_i X_i\right)\left(\sum_i Z_i\right)\left(\sum_i Y_i\right) - \left(\sum_i X_i\right)\left(\sum_i Y_i\right)\left(\sum_i Z_i\right).$$

[where terms multiplying to zero have been included for a compact formula]

Claim: For any finite abelian group $G \subset SL_3(\mathbb{C})$, (Q_G, W_G) yields quotient of hexagonal lattice as its bipartite tiling.

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Firstly, by construction each vertex x_i has three arrows incident to it which are outgoing, x_i, y_i & z_i.

The relations in $I_{W_G} = \partial W_G$ look like

$z_j y_k - y_j z_l$ (illustrating differentiation by arrow $x_i \xrightarrow[x_i]{} x_j$)

such that y_j, z_j are the associated outgoing arrows from vertex x_j,

k chosen as the target of arrow z_j

& l chosen as the target of arrow y_j.

Note that

$$\begin{tikzcd}
 & & x_k = (x_i\otimes x_a)\otimes x_c \arrow[dr, "y_k"] & \\
x_i \arrow[r, "x_i"] & x_j = x_i\otimes x_a \arrow[ur, "z_j"] \arrow[dr, "y_j"] & & x_i\otimes x_a\otimes x_b\otimes x_c = x_i \text{ (since } a+b+c\equiv 0 \bmod m) \\
 & & x_l = (x_i\otimes x_a)\otimes x_b \arrow[ur, "z_l"] &
\end{tikzcd}$$

so the products making up the terms of W_G are indeed 3-cycles as long as the sources and targets match up in the middle.

Further, each vertex x_i has three incoming arrows from $x_i \otimes x_a^*$, $x_i \otimes x_b^*$, and $x_i \otimes x_c^*$.

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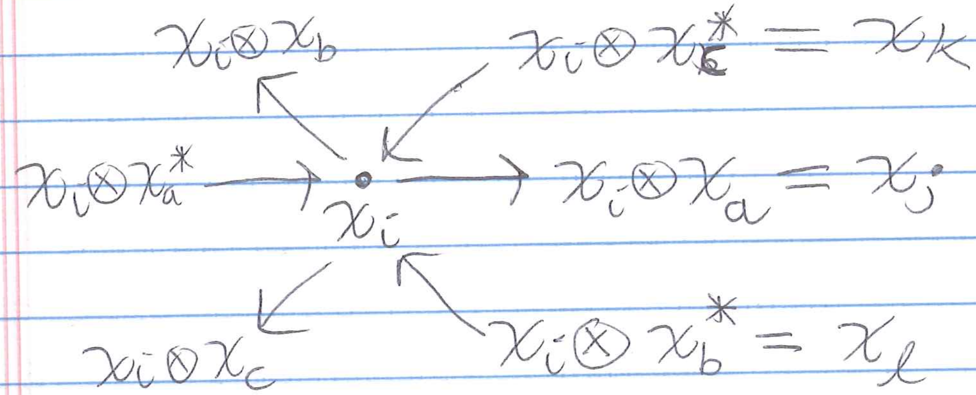
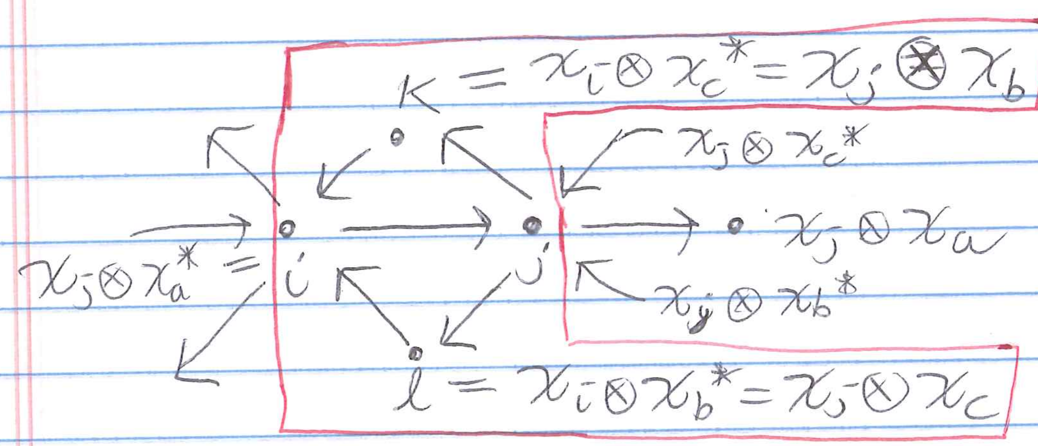
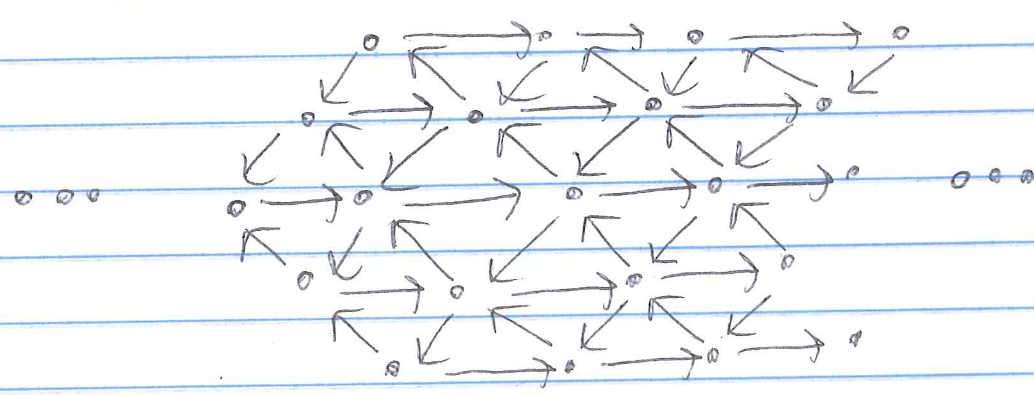
Let us unfold quiver Q_G and note that each vertex locally looks like



and by the relations of ∂W_G, we glue these together as



So we get oriented triangle lattice



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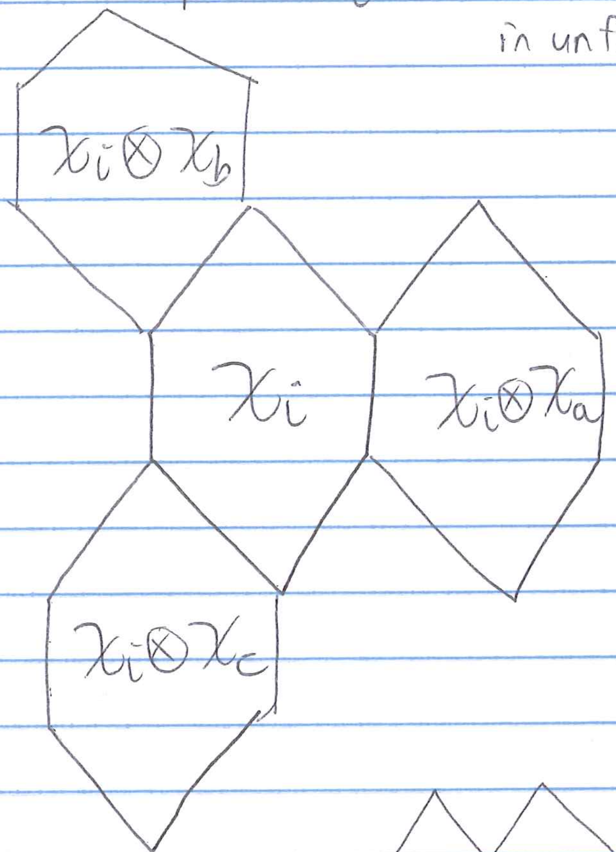
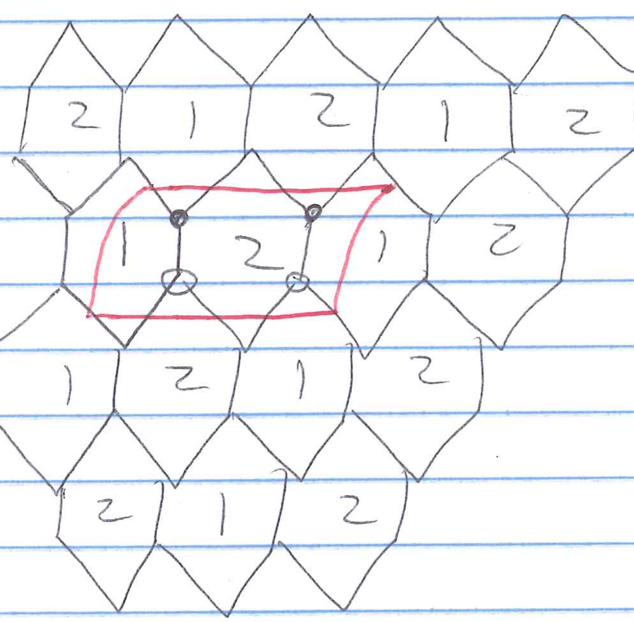
(9) Planar dual is hexagonal bipartite tiling.

irred. rep. $\chi_i \longleftrightarrow$ vertex i in unfolded quiver $\longleftrightarrow$ face i in hexagonal bipartite tiling



e.g. $\mathbb{Z}_2$



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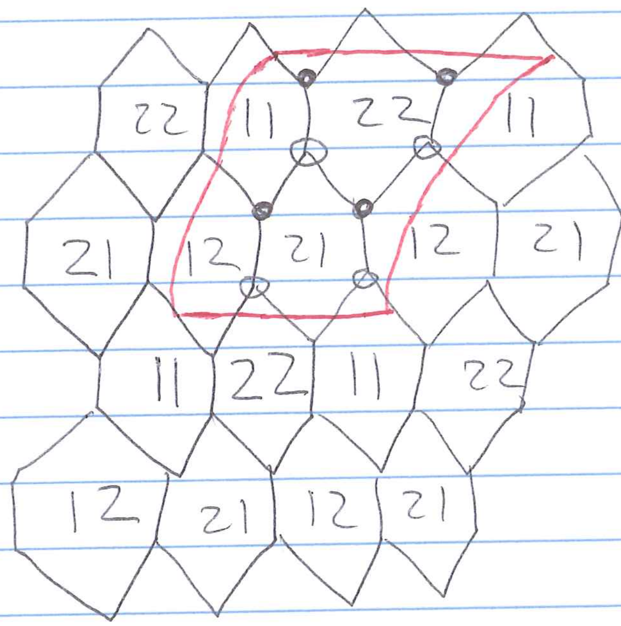
e.g. $\mathbb{Z}_2 \times \mathbb{Z}_2$



<u>Exercise:</u> $\mathbb{Z}_3(1,1,1)$ and $\mathbb{Z}_3(2,1,0)$.

<u>Claim:</u> Superpotential Algebra $\mathbb{C}Q_G/\partial W_G = \mathbb{C}[x,y,z] \rtimes G$

(Anyone tried Exercise from last time for $G = \mathbb{Z}_2$? i.e. Problem 3a on HW2)

Rem: $\boxed{\text{Errata}}$ from Lecture 15: Should have let x, y, z be liner sums of arrows (rather than sums of 2-paths)

In fact, let $X = \sum_i x_i$ (corresponding to tensoring w/ x_a)

Y, Z defined analogously so that $x_{nat} = x_a \oplus x_b \oplus x_c$.

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$$XY = \left(\sum_i x_i\right)\left(\sum_j y_j\right)$$

$$= \sum x_i \, y_{j_{i,a}}$$

$$x_i \to x_i \otimes x_a \to x_i \otimes x_a \otimes x_b = x_i \otimes x_c^*$$

$$x_i \otimes x_a = x_{j_{i,a}}$$

Analogously, $yx = \sum y_i \, x_{j_{i,b}}$

$$x_i \to x_i \otimes x_b \to x_i \otimes x_b \otimes x_a = x_i \otimes x_c^*$$

But these two-paths guaranteed to be equal modulo ∂W_G by considering differentiation by the appropriate z_K's

$$x_K = x_i \otimes x_c^* \xrightarrow{z_K} x_i$$

Using multiple relations potentially, we obtain $xy = yx$.

The proof that $xz = zx$ and $yz = zy$ are analogous.

Letting $x \leftrightarrow (x, id)$, $y \leftrightarrow (y, id)$, $z \leftrightarrow (z, id)$ in $\mathbb{C}[x, y, z] \rtimes G$, we exactly the desired commutation relations.

We next need to define $(p(x,y,z), g)$'s for arbitrary $g \in G$.

We begin by defining $(1, g)$ as $\boxed{\sum_{i=1}^{m} e_i \, x_i(g)}$ $\in \mathbb{C}Q_G$

where e_i = lazy path of length 0 $i \circlearrowleft$

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e.g. $\mathbb{Z}_2$ $\quad \underset{\mathbb{C}[x,y,z]}{\underset{\cap}{(1}}, \underset{\mathbb{Z}_2}{\underset{\in}{\varepsilon}}) = e_0 \quad e_1$ $\quad [-1 = \mathrm{sgn}(\varepsilon)]$

e.g. $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad (1, (\varepsilon_1, \varepsilon_2)) = e_{00} - e_{10} - e_{01} + e_{11}$

e.g. $\mathbb{Z}_3 (1,1,1)$ $\quad (1, \sigma) = e_0 + \rho e_1 + \rho^2 e_2$

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we then define $(P(x,y,z), g)$ as

$$(P(x,y,z), 1) \cdot (1, g)$$

where $(x,1), (y,1), (z,1)$ defined,
for an arbitrary polynomial P in $\mathbb{C}[x,y,z]$,
we multiply or sum appropriately to get $(P(x,y,z), 1)$.

Then multiplication by $(1, g)$ scales arrows accordingly based on their source.

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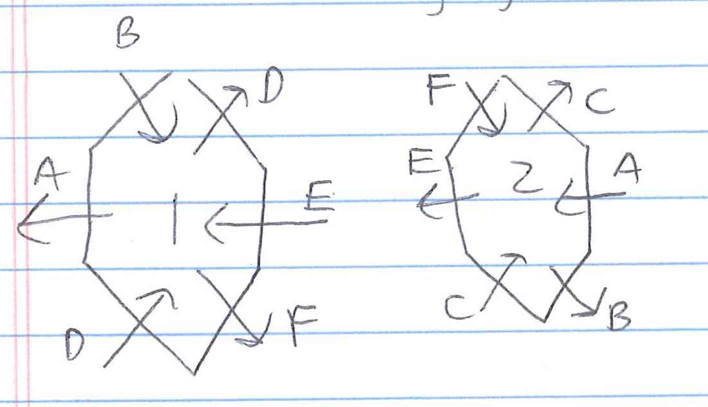
e.g. $\mathbb{Z}_2$ $\quad (x,1) = A + E, \quad (y,1) = D + C, \quad (z,1) = F + B$
but $(x, \varepsilon) = A - E \quad (y, \varepsilon) = D - C \quad (z, \varepsilon) = F - B$



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we thus see

$$(1, g_1)\circ(1, g_2) = \left(\sum_i e_i x_i(g_1)\right)\left(\sum_j e_j x_j(g_2)\right)$$

(since $e_i^2 = e_i$, $e_i e_j = 0$ if $i \neq j$)

$$= \sum_i e_i x_i(g_1) x_i(g_2)$$

$$= \sum_i e_i x_i(g_1 g_2) = (1, g_1 g_2)$$

$$(1, g_1)\circ(Q(x,y,z), 1) =$$

$$\left(\sum_i e_i x_i(g_1)\right)\cdot Q\left(\sum_j x_j, \sum_k y_k, \sum_l z_l\right)$$

$$\|$$

(again, $e_i x_j = x_j \Leftrightarrow i = j$ etc.)

$$Q\left(\sum_i x_i(g_1) x_i, \sum_i x_i(g_1) y_i, \sum_i x_i(g_1) z_i\right)$$

interchanging source & target

$$= {}^{g_1}Q(x,y,z)\left(\sum_j e_j x_j(g_1)\right) = {}^{g_1}Q(x,y,z)\circ(1, g_1)$$

Thus $(P(x,y,z), g_1)\circ(Q(x,y,z), g_2) =$

$$(P(x,y,z),1)\circ(1, g_1)\circ(Q(x,y,z),1)\circ(1, g_2) =$$

$$(P(x,y,z),1)\circ({}^{g_1}Q(x,y,z),1)\circ\underbrace{(1,g_1)\circ(1,g_2)}_{(1,g_1 g_2)}$$

$$= \boxed{(P\,{}^{g_1}Q, g_1 g_2)}$$

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14 In conclusion, we have a homomorphism

$$\mathbb{C}[x,y,z] \rtimes G \longrightarrow \mathbb{C}Q_G / \partial W_G$$

so that the images $(p(x,y,z), g)$ satisfy the necessary relations of the skew group ring.

By construction, the images also satisfy the relations of ∂W_G. (this was how we ensured commutativity of the x, y, z's)

<u>To show this homomorphism is an isomorphism</u> we show that the idempotents e_i of $\mathbb{C}Q_G/\partial W_G$ can all be generated from $\mathbb{C}$-linear combinations of $(1,g)$'s.

e.g. $\mathbb{Z}_2$ $(1, \mathrm{id}) = e_1 + e_2$, $(1, \varepsilon) = e_1 - e_2$.

$\boxed{\text{Ideas?}}$

<u>Answer:</u> orthogonality of characters

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$$

$$\sum_{g \in G} c_g (1,g) = \sum_{g \in G} c_g \sum_i \chi_i(g) e_i = \sum_i e_i \sum_{g \in G} c_g \chi_i(g)$$

Thus, for each j, let c_g's be defined as $c_g = \overline{\chi_j(g)}$.

Then $\boxed{\sum_{g \in G} \overline{\chi_j(g)} (1,g) = e_j}$

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(15) Thus the idempotents $\{e_1, \dots, e_m\}$ of $\mathbb{C}Q_G / \partial W_G$ indeed in $\mathbb{C}[x,y,z] \rtimes G$ under the inverse image.

$e_i X = X_i$, the specific arrow outgoing from vertex $\chi_i \longrightarrow \chi_i \otimes \chi_a$

$e_i Y$, $e_i Z$ are the other two outgoing arrows from vertex χ_i

and repeating for every vertex i, we thus have the entire superpotential algebra.

when G abelian $\blacksquare$

We thus have the following schematic