4/13/15 Lecture 22: Extremal versus External perf. matchings

Geometric consistency versus (R-symm.) consistency

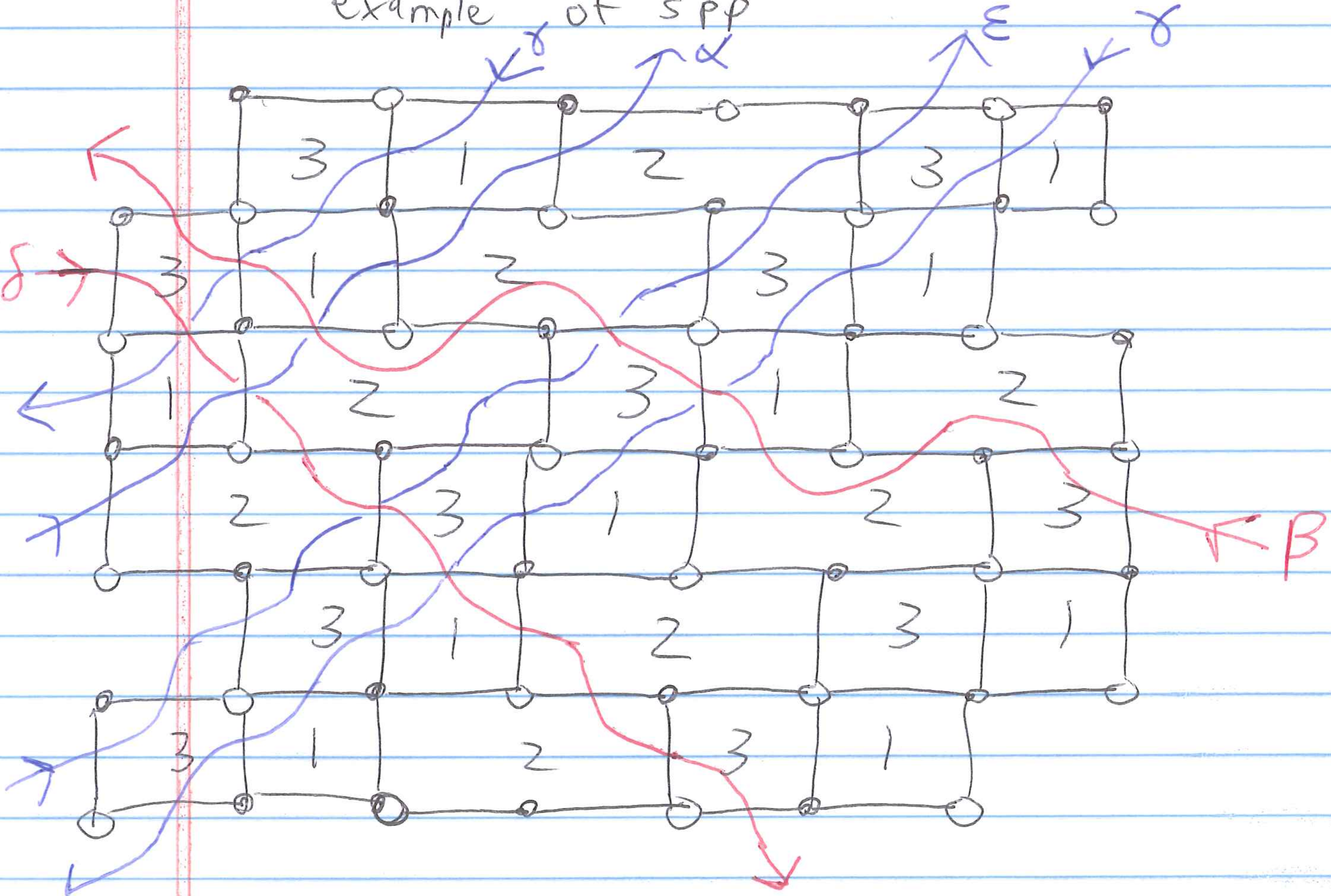
Geometric Consistency (Broomhead, Prop 3.12)

a) Any zig-zag path has no self-intersections

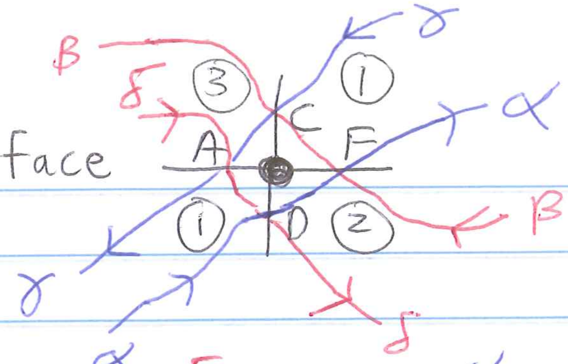
b) If $[\alpha]$ & $[\beta]$ in $H_1(Y;\mathbb{Z})$ linearly independent, then zig-zag paths α & β intersect in exactly one arrow.

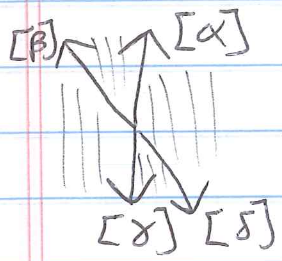
c) If $[\alpha]$ & $[\beta]$ are linearly dependent then zig-zag paths α & β do not intersect.

We build local & global zig-zag fans & $P(\sigma)$'s for example of SPP

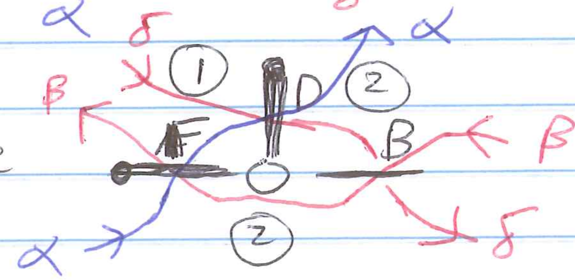


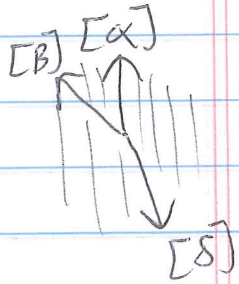
4/13/15 ② ZZ's intersecting face



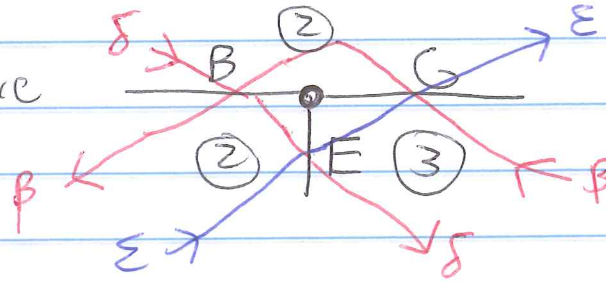


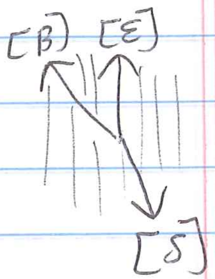
ZZ's intersecting Face



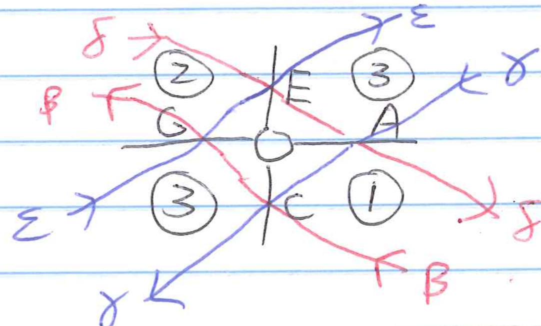


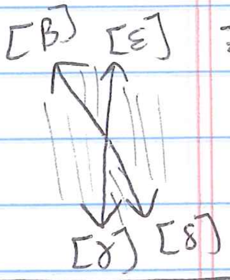
ZZ's intersecting Face



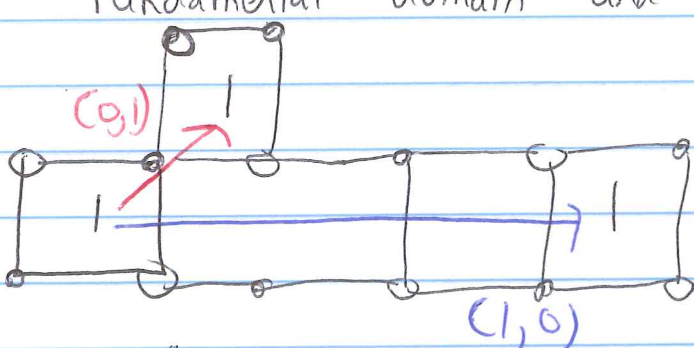


ZZ's intersecting Face





using Fundamental domain and coordinates



$[\alpha] = (0,1)$

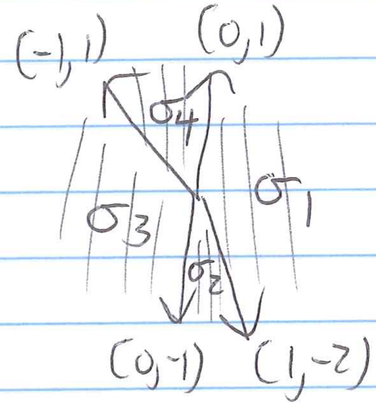
$[\beta] = (-1,1)$

$[\gamma] = (0,-1)$

$[\delta] = (1,-2)$

$[\varepsilon] = (0,1)$

4/13/15 ③ $\Rightarrow$ SPP has Global Zig-zag fan



For σ_i a 2-dim cone in Global zig-zag fan,

$$P(\sigma_i) = \frac{1}{2}\sum_{F \in Q_2} P_F(\sigma_i)$$

e.g.

$$P(\sigma_1) = \frac{1}{2}\left[D + D + E + E\right]$$

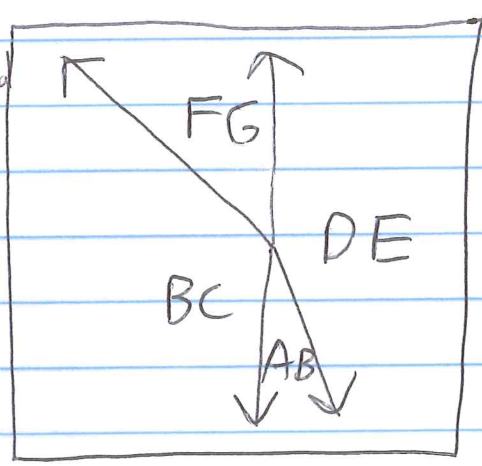
$$P(\sigma_2) = \frac{1}{2}\left[A + B + B + A\right]$$

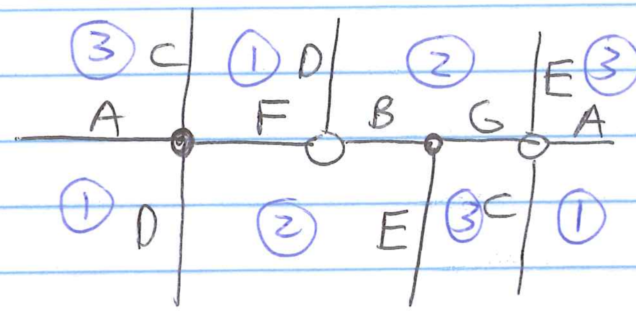
$$P(\sigma_3) = \frac{1}{2}\left[C + B + B + C\right]$$

$$P(\sigma_4) = \frac{1}{2}\left[F + F + G + G\right]$$

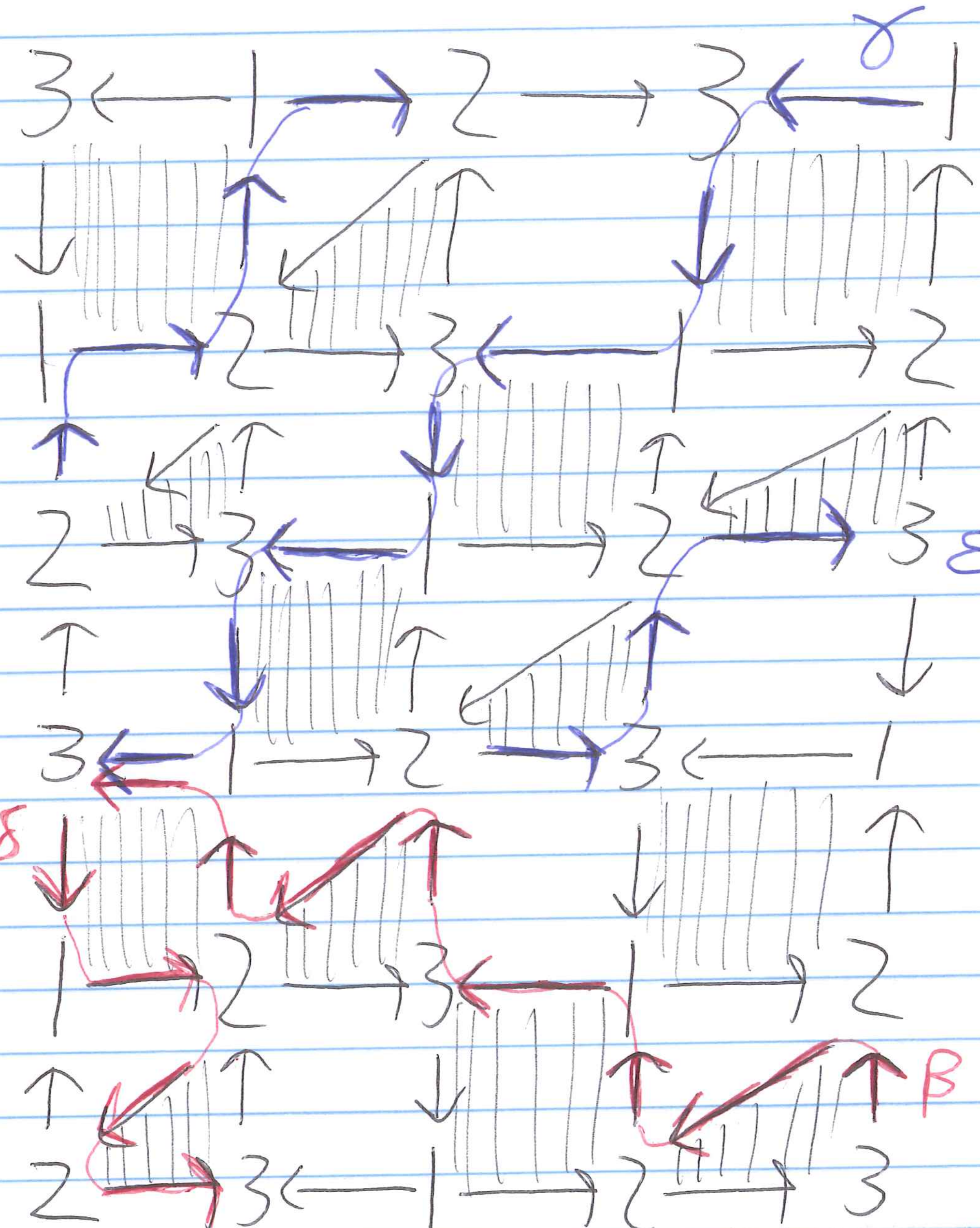
two missing non-extremal perfect matchings are

DG & EF





4/13/15 (4) dual picture of this dimer model (as unfolded quiver)



Let us consider black & white boundary flows:

e.g., $B(\alpha) = \gamma$, $W(\alpha) = \cdots \swarrow z \swarrow z \swarrow z \swarrow \cdots$

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$B(\alpha) = $
$$\begin{tikzcd} & 3 & 1 \arrow[l, "C"'] \arrow[d, "A"] \\ 3 & 1 \arrow[l, "C"'] & \end{tikzcd}$$

$W(\alpha) = $
$$\begin{tikzcd} & & 2 \arrow[dl, "B"'] \\ & 2 \arrow[dl, "B"'] & \\ 2 & & \end{tikzcd}$$

$B(\beta) = $
$$\begin{tikzcd} 3 \arrow[d, "A"'] & & \\ 1 \arrow[r, "D"] & 2 \arrow[r, "E"] & 3 \end{tikzcd}$$

$W(\beta) = $
$$\begin{tikzcd} 1 \arrow[r, "D"] & 2 \arrow[r, "E"] & 3 \arrow[d, "A"] \\ & & 1 \end{tikzcd}$$

$B(\gamma) = $
$$\begin{tikzcd} & 1 \arrow[r, "D"] & 2 \\ 1 \arrow[r, "D"] & 2 \arrow[u, "F"] & \end{tikzcd}$$

$W(\gamma) = $
$$\begin{tikzcd} & 2 \\ 2 \arrow[r, "E"] & 3 \arrow[u, "G"] \\ 3 \arrow[u, "G"] & \end{tikzcd}$$

$B(\delta) = $
$$\begin{tikzcd} 3 & 1 \arrow[l, "E"'] \\ & 2 \arrow[u, "F"'] \\ & 3 \arrow[u, "G"'] \end{tikzcd}$$

$W(\delta) = $
$$\begin{tikzcd} 2 & \\ 3 \arrow[u, "G"] & 1 \arrow[l, "C"'] \\ & 2 \arrow[u, "F"'] \end{tikzcd}$$

$B(\varepsilon) = $
$$\begin{tikzcd} & & 2 \arrow[dl, "B"'] \\ & 2 \arrow[dl, "B"'] & \\ 2 & & \end{tikzcd}$$

$W(\varepsilon) = $
$$\begin{tikzcd} & 3 \arrow[d, "A"'] \\ 3 & 1 \arrow[l, "C"'] \\ 1 \arrow[from=u, "A"'] & \end{tikzcd}$$

4/13/15 (6) Given a Global Zig-Zag Fan for a dimer model, with two dimensional cones $\{\sigma_1, \ldots, \sigma_m\}$.

Let $(\alpha_1, \ldots, \alpha_m)$ be a set of representative zig-zag paths such that σ_i = defined by rays $[\alpha_i]$, $[\alpha_{i+1}]$ (subscripts mod m)

<u>Claim:</u>

- If M is an <u>internal</u> perfect matching, M evaluates positively on $B(\alpha_i)$ and $W(\alpha_i)$.
- If M is an <u>external</u> (but not extremal) perfect matching, $\exists!\ \alpha_i$ s.t. M evalues to zero on $B(\alpha_i)$ & $W(\alpha_i)$.
- If M is an <u>extremal</u> perfect matching, M evaluates to zero on $B(\alpha_i)$ & $W(\alpha_i)$ for exactly two α_i's.

<u>SPP e.g.</u> (no internal perfect matchings)

$AB \to \gamma, \delta$

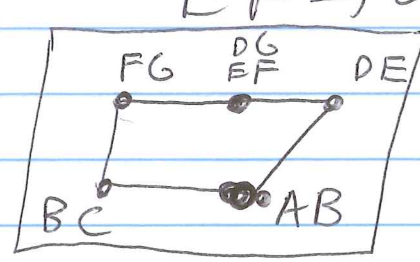
$DG \to \alpha$

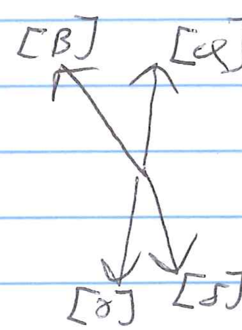
$DE \to \delta, \alpha$

$EF \to \alpha$

$FG \to \alpha, \beta$

$BC \to \beta, \gamma$

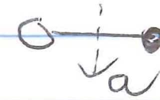




Recording which zig-zags out of $\{\alpha, \beta, \gamma, \delta\}$ are zero of these perfect matchings

4/13/15 7 Broomhead, Lemma 4.19 $P(\sigma) := \frac{1}{2} \sum_{F \in Q_2} P_F(\sigma)$ is a perfect matching.

PF: Any arrow a is in the boundary of exactly two faces, i.e. $\{F_W, F_B\}$



$$P(\sigma)(a) = \frac{1}{2} P_{F_B}(\sigma)(a) + \frac{1}{2} P_{F_W}(\sigma)(a)$$

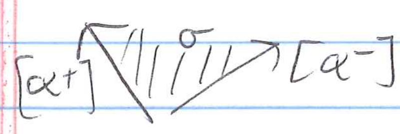
Let α^+ and α^- be the two zig-zag paths through arrow a.

Let $\mathcal{E}(F_B)$ and $\mathcal{E}(F_W)$ be the corresponding local zig-zag fans of those faces.

$\mathcal{E}(F_B)$ and $\mathcal{E}(F_W)$ both contain rays generated by $[\alpha^+]$ & $[\alpha^-]$.

As discussed last class, cannot be another ray $[\beta]$ between $[\alpha^+]$ and $[\alpha^-]$ in $\mathcal{E}(F_B)$ nor $\mathcal{E}(F_W)$.

$\Rightarrow$ $\mathcal{E}(F_B)$ & $\mathcal{E}(F_W)$ both contain the 2-dim cone



determined by the arrow a

$\Rightarrow$ $P_{F_B}(\sigma)(a) = P_{F_W}(\sigma)(a) = 1$

$P_{F_B}(\sigma)(a') = 0$ for any other $a' \neq a \in \partial F_B$

$P_{F_W}(\sigma)(a'') = 0$ for any other $a'' \neq a \in \partial F_W$.

Thus, for each F_B, F_W, $\exists!$ arrow $a \in \partial F$ on which $P(\sigma)$ evaluates to 1. Summing over all $F \in Q_2$, $P(\sigma)$ is a P. M.

4/13/15 <u>Def:</u> A dimer model is called <u>consistent</u> if there exists an R-symmetry satisfying

$$\sum_{\substack{a \text{ inc.} \\ \text{to } v}} R_a = \deg(R)\left(\frac{\#\text{ arrows incid. to } v}{2} - 1\right) \forall v.$$

<u>Claim:</u> If a dimer model is consistent and

$$\sum_{a \in \partial f} R_a = \deg(R) \quad \forall f \in Q_2, \text{ then}$$

the dimer model must correspond to a <u>quiver on a torus</u>.

<u>Pf:</u> $$(\deg R)\,|Q_2| = 2\sum_{\substack{f \in Q_2 \\ a \in \partial f}} R_a = 2\sum_{\substack{v \in Q_0 \\ a \text{ inc. to } v}} R_a$$

$$= \deg(R)\left(|Q_1| - |Q_0|\right) \Rightarrow |Q_2| - |Q_1| + |Q_0| = 0$$

$\Rightarrow$ Euler characteristic $= 0$. ▨

<u>Rem:</u> Y being klein bottle might also be possible if we did not require Y orientable.

<u>Rem:</u> Physicists would phrase these conditions slightly differently:

- $\sum_{a \in \partial f} R_a = 2 \quad \forall f \in Q_2$
- $\sum_{a \text{ inc. to } v} (1 - R_a) = 2 \quad \forall v \in Q_0$

[assuming R-symmetries degree 2 but in $\mathbb{R}^{Q_1}$ instead of $\mathbb{Z}^{Q_1}$.]

<u>related to vanishing of β-functions, superconformal invariance.</u>

4/13/15 (9)

Recall that an <u>R-symmetry</u> is a global symmetry that acts with strictly positive weights on all arrows.

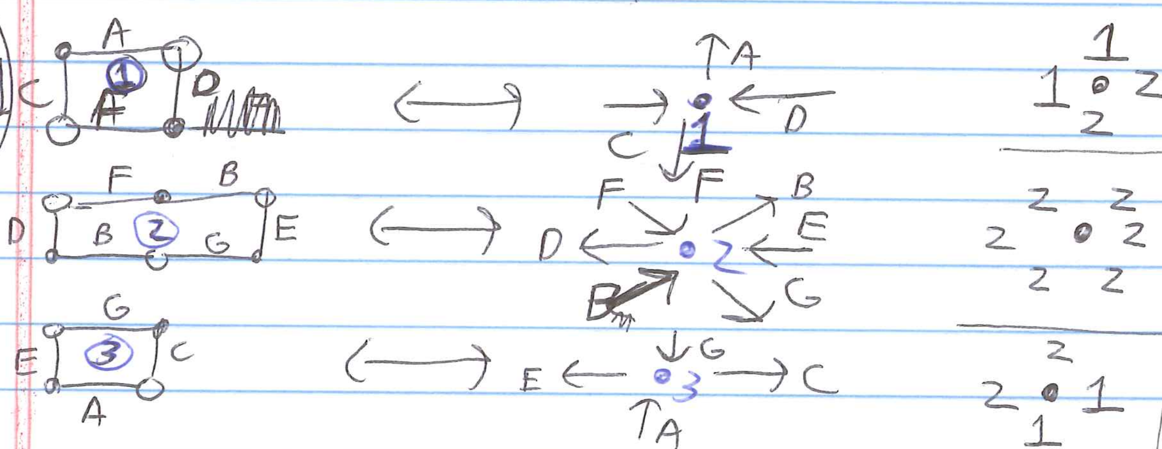
We say that an R-symmetry is <u>anomaly-free</u> if for every vertex of the quiver

$$\sum_{\substack{\text{incident arrows} \\ a \text{ to } v}} R_a = \deg(R)\left(\frac{\#\text{ arrows incident to } v}{2} - 1\right)$$

e.g. SPP, adding all perfect matchings together gives global symmetry $R \in \mathbb{Z}^{Q_1}$ defined as

$R(A)=1,\ R(B)=2,\ R(C)=1,\ R(D)=2,\ R(E)=2,\ R(F)=2,$
$R(G)=2$

which is an R-symmetry.

$\left(\frac{\#\text{ arrows inc. to } v}{2} - 1\right)$		$\sum R_a$
1		6
2		12
1		6

deg R = 6 (since the sum of six perfect matchings)

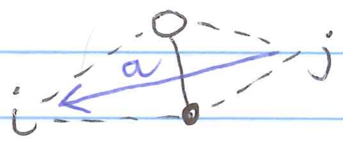
Thus, this R is indeed <u>anomoly-free</u>.

4/13/15 (10)

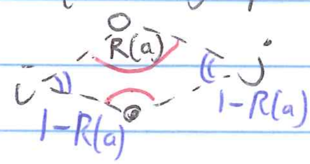
We can use geometry to show that a <u>geometrically consistent</u> dimer model is <u>consistent</u>:

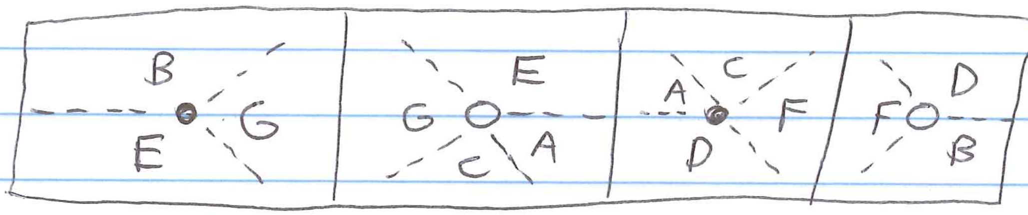
Normalizing so that $\deg R = 2\pi$ $\left(\begin{array}{c}\text{multiply through by}\\ 2\pi/\deg R\end{array}\right)$

we draw dimer model so quad graph has angles



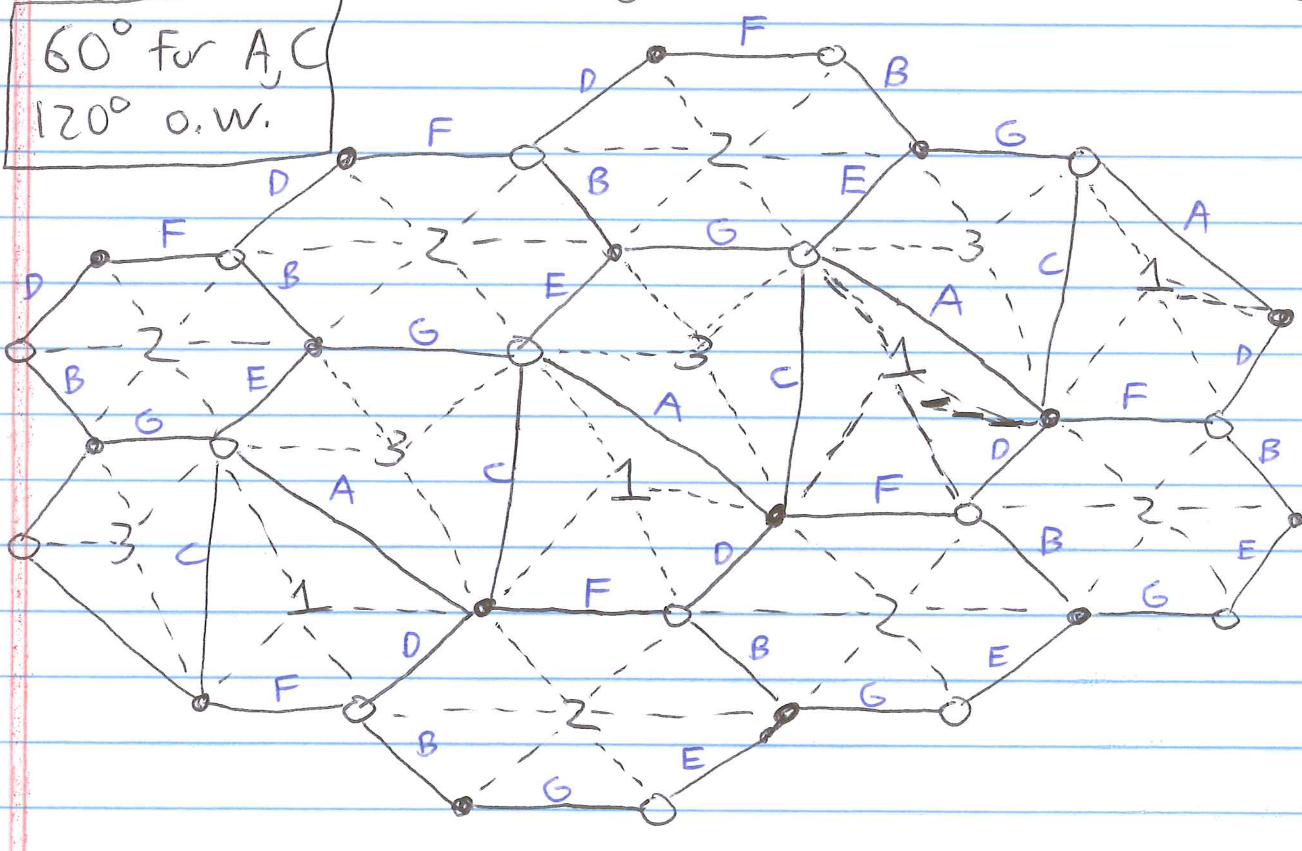
so





<u>e.g. SPP</u> $R(A)=R(C)=\frac{\pi}{3}$, $R(B)=R(D)=R(E)=R(F)=R(G)=\frac{2\pi}{3}$

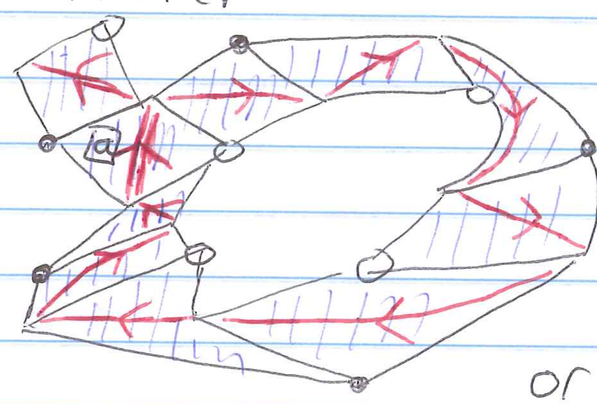
$\boxed{\begin{array}{l}60^\circ \text{ for } A, C\\ 120^\circ \text{ o.w.}\end{array}}$



4/13/15 (11) In general, suppose we had a geom. consistent dimer model (conditions on zig-zag paths) and build the corresponding quad graph and train tracks.

We wish to show that such train tracks have
- no self-intersections
- no two train tracks intersect more than once

Suppose otherwise.

(1) 

self-intersecting train track must correspond to self-intersecting zig-zag path or possible lifts to two zig-zag paths, but both using arrow a and with same homology. Contradicts $[\alpha]=[\beta]$
$\Rightarrow$ no intersections.

(2) The lift of a train track is a zig-zag path in this way so double-intersection of TT would imply double-intersection of ZZ's

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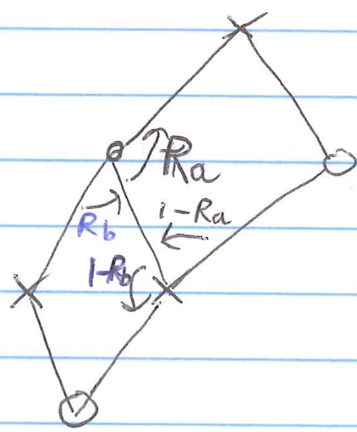
Secondly, with these conditions on train tracks in the quad graph, we claim the quad graph has a rhombic embedding on the torus meaning all line segments of quad graph have the same length.

4/13/15 (12) In fact <u>rhombic embedding</u> <u>of quad graph</u> $\iff$ train tracks geom. consistent

Proven in [Kenyon-Schlenker] (Theorem 5.1)

So <u>Geom. consistent</u> $\iff$ <u>rhombic embedding</u> $\Rightarrow$ <u>consistent</u>

where the last implication follows by the fact that rhombi can be put together as part of an embedding $\Rightarrow$

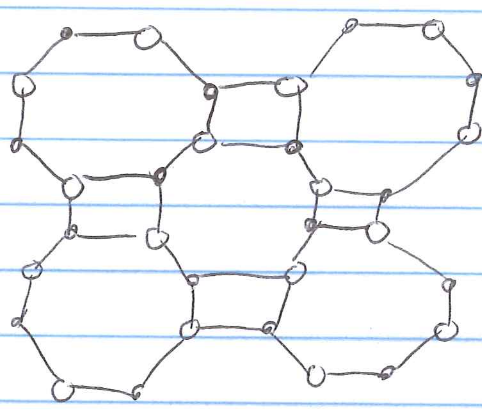


$$\sum_{a \in \partial F} R_a = 2\pi$$

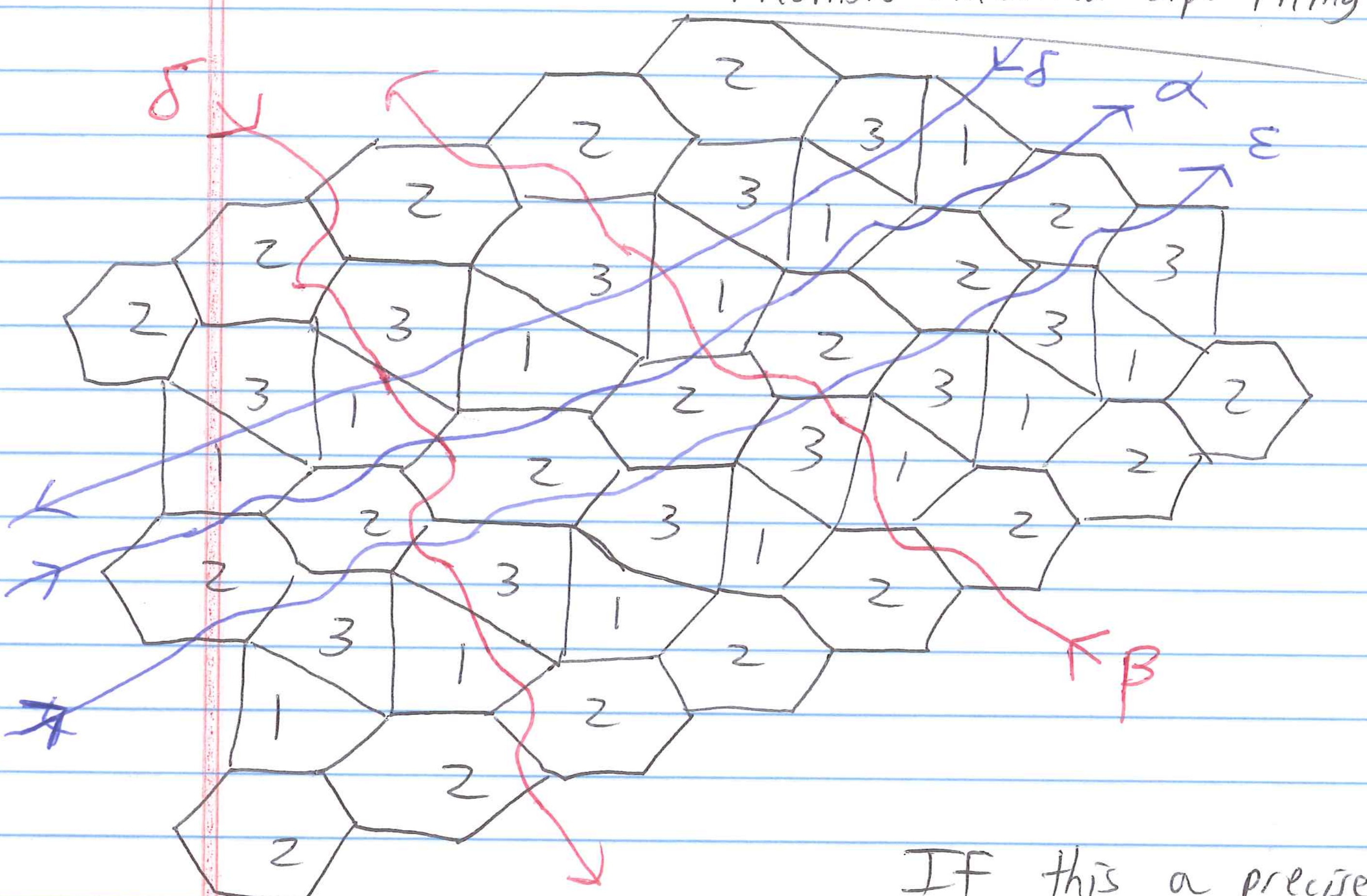
$$\sum_{a \in \partial v} (1 - R_a) = 2\pi$$

<u>Note:</u> consistent $\not\Rightarrow$ Geom. consistent

Broomhead <u>e.g 3.4</u>



4/13/15 13 Zig-Zags for SPP redrawn on rhombic-embedded bip. tiling



If this a precise rhombic embedding, these zig-zag paths should really become straight lines of appropriate slopes.

$[\alpha] = (0,1)$
$[\beta] = (-1,1)$
$[\gamma] = (0,-1)$
$[\delta] = (1,-2)$
$[\varepsilon] = (0,1)$

Also called an isoradial embedding.

Compare with output of Goncharov-Kenyon algorithm.

Primitive vectors on boundary of toric diag $\Delta \longleftrightarrow$ homology loops $\longleftrightarrow$ zig-zag paths